An Overview of Moreau's Sweeping Processes with Applications

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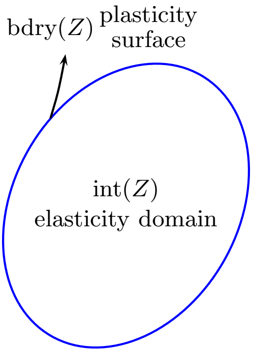
Outline

First example: the Prandtl-Reuss elastoplastic model

Let Z be a closed and convex set of the dimensional vector space E of symmetric tensors $N \times N$ with $\mathrm{int}(Z) \neq \emptyset$.



- We apply a time-dependent strain $\varepsilon(t)$. We would like to know the stress $\sigma(t)$ applied to the body Z, that is, the map

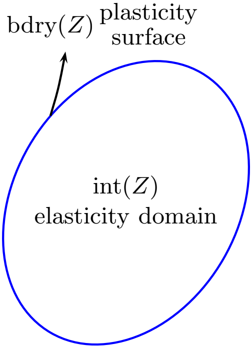
$$\varepsilon(t) \mapsto \sigma(t).$$

- $\varepsilon(t)$ is the time-dependent *strain tensor*.
- $\sigma(t)$ is the time-dependent *stress tensor*.

[6] P. Krejči: Vector hysteresis models. Eur. J. Appl. Math. 2, 281-292 (1991).

First example: the Prandtl-Reuss elastoplastic model

Let Z be a closed and convex set of the dimensional vector space E of symmetric tensors $N \times N$ with $\text{int}(Z) \neq \emptyset$.



Assumptions:

- The strain tensor $\varepsilon = \varepsilon^e + \varepsilon^p$, where ε^e is the elastic strain and ε^p is the plastic strain.
- The elastic strain ε^e is related to the stress tensor σ linearly, that is, $\varepsilon^e = A^2\sigma$, where A is a (constant) symmetric positive definite matrix.
- The stress tensor cannot leave the domain, that is, $\sigma(t) \in Z$ for all $z \in Z$
- The principle of maximal dissipation holds:

$$\langle \dot{\varepsilon}^p(t), z \rangle \leq \langle \dot{\varepsilon}^p(t), \sigma(t) \rangle \text{ for all } z \in Z,$$

[6] P. Krejči: Vector hysteresis models. Eur. J. Appl. Math. 2, 281-292 (1991).

First example: the Prandtl-Reuss elastoplastic model

Therefore, we arrive to the following differential variational inequality:

$$\left\langle -A\dot{\sigma}(t) + A^{-1}\dot{\varepsilon}(t), Az - A\sigma(t) \right\rangle \leq 0 \quad \text{for all } z \in Z.$$

Setting

$$x(t) := A\sigma(t) - A^{-1}\varepsilon(t) \text{ and } C(t) := -A^{-1}\varepsilon(t) + A(Z),$$

we arrive to the differential inclusion:

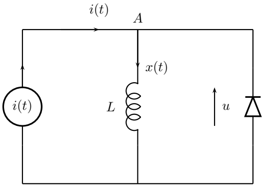
$$\dot{x}(t) \in -N\left(C(t); x(t)\right),$$

where $N(C(t); \cdot)$ denotes the convex normal cone to the set $C(t)$.

[6] P. Krejči: Vector hysteresis models. Eur. J. Appl. Math. 2, 281-292 (1991).

Second example: nonsmooth electrical circuits

Let us consider a circuit with an ideal diode, an inductor and a current source $i(t)$:



If x denotes the current through the inductance, the dynamics is given by

$$\begin{cases} \dot{x}(t) = u(t) \\ \mathbb{R}_+ \ni x(t) - i(t) \perp u(t) \in \mathbb{R}_+. \end{cases} \tag{1}$$

Second example: electrical circuits

The second line in (1) can be written as

$$\mathbb{R}_+ \ni x(t) - i(t) \perp u(t) \in \mathbb{R}_+ \quad \Leftrightarrow \quad u(t) \in -N\left(\mathbb{R}_+; x(t) - i(t)\right).$$

Therefore, the system (1) is equivalent to the following differential inclusion:

$$\dot{x}(t) \in -N\left(C(t); x(t)\right),$$

where $C(t) := [c(t), +\infty[$.

Moreau's sweeping process

In these two examples, the motion can be described by the so-called *Sweeeping Processes*:

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); x(t)\right) & \text{a.e. } t \in [0,T], \\ x(0) = x_0 \in C(0), \end{cases}$$

where $C(t)$ is a closed set for all $t \in [0,T]$.

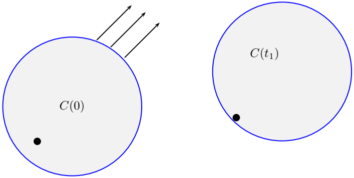
- The sweeping process was introduced by J.-J Moreau in 1971[2] to model an elastoplastic mechanical system.
- The sweeping process appears in several fields such as nonsmooth electrical circuits, nonsmooth mechanics, crowd motion, hysteresis phenomena, etc.

[2][8] J.-J. Moreau: Rafle par un convexe variable I. Sém. Anal. Convexe Montpellier (1971), Exposé 15.

Interpretation of the sweeping process

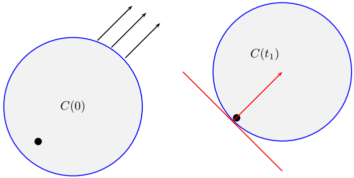
Consider a large ring that contains a small ball. The ring will start to move at time $t = 0$.

Depending on the motion of the ring, the ball will just stay where it is (in case it is not hit by the ring), or otherwise it is swept towards the interior of the ring.



Interpretation of the sweeping process

Consider a large ring that contains a small ball, and the ring will start to move at time $t = 0$.
Depending on the motion of the ring, the ball will just stay where it is (in case it is not hit by the ring), or otherwise it is swept towards the interior of the ring.
In this latter case the velocity of the ball has to point inwards to the ring in order not to leave.



Moreau's sweeping process

Main difficulties:

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); x(t)\right) & \text{a.e. } t \in [0, T], \\ x(0) = x_0 \in C(0), \end{cases} \tag{1.1}$$

where $C(t)$ is a closed set for all $t \in [0, T]$.

1. The right-hand side of (1.1) is unbounded.
2. The problem (1.1) is a constrained differential inclusion.
3. For *discontinuous* moving sets, what is a good notion of solution?

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Moreau's sweeping process

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); x(t)\right) & \text{a.e. } t \in [0, T], \\ x(0) = x_0 \in C(0), \end{cases}$$

where $C(t)$ is a closed set for all $t \in [0, T]$.

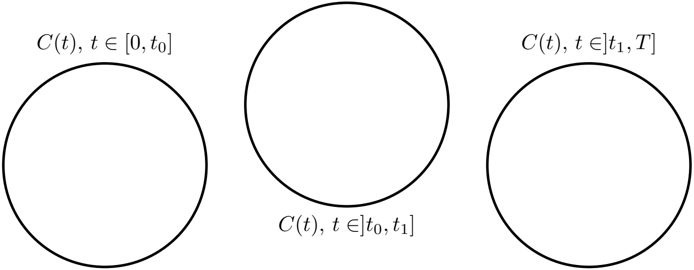

Figure: Sweeping processes without continuous solution. $0 < t_0 < t_1 < T$.

First basic existence result

Theorem (Moreau 1971 [8, 9])

If the sets $(C(t))_{t\geq 0}$ are nonempty, closed and convex with

$$\text{Haus}(C(t), C(s)) = \max\{\sup_{x\in C} d_{C(t)}(x), \sup_{x\in D} d_{C(s)}(x)\} \leq \kappa|t-s|,$$

for some $\kappa \geq 0$. Then, there exists a unique Lipschitz solution of the sweeping process:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) & a.e.\ t \in [0,T]. \\ x(0) = x_0 \in C(0). \end{cases} \tag{SP}$$

Moreover, $\|\dot{x}(t)\| \leq \kappa$ for a.e. $t \in [0,T]$.

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[8] J.J. Moreau: Rafle par un convexe variable I. Sém. Anal. Convexe Montpellier (1971), Exposé 15.

[9] J.J. Moreau: Rafle par un convexe variable II. Sém. Anal. Convexe Montpellier (1972), Exposé 3.

Proof: The catching-up algorithm

Let $\pi_n = \{t_0^n, \ldots, t_n^n\}$ be a partition of $[0, T]$. Over every interval $]t_i^n, t_{i+1}^n]$ we use the implicit discretization :

$$\dot{x}(t) \approx \frac{x_{i+1}^n - x_i^n}{t_{i+1}^n - t_i^n}$$
$$N\left(C(t); x(t)\right) \approx N\left(C(t_{i+1}^n); x_{i+1}^n\right).$$

Therefore, for all $i = 0, \ldots, n-1$

$$x_i^n \in \left(I + N_{C(t_{i+1}^n)}(\cdot)\right)\left(x_{i+1}^n\right)$$

Hence, we arrive to

$$\begin{cases} x_{i+1}^n = \text{proj}_{C(t_{i+1}^n)}(x_i^n) & \text{"Catching-up algorithm"}, \\ x_0^n = x_0 \in C(0). \end{cases}$$

Proof: The catching-up algorithm

$$\begin{cases} x_{i+1}^n = \text{proj}_{C(t_{i+1}^n)}(x_i^n) & \text{"Catching-up algorithm"}, \\ x_0^n = x_0 \in C(0). \end{cases}$$

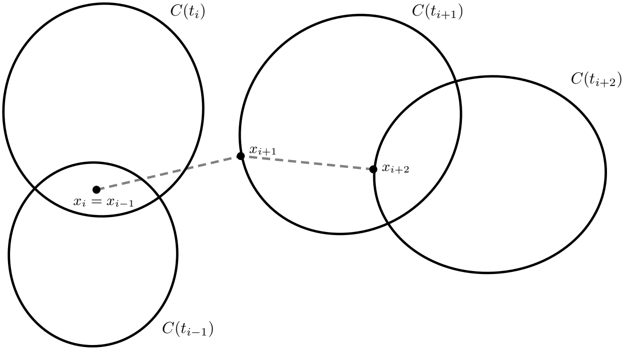

Figure: The Catching-up algorithm

Proof: The catching-up algorithm

It is possible to prove that:

- $d_{C(t_{i+1}^n)}(x_i^n) \leq \kappa|t_{i+1}^n - t_i^n|$ for $0 \leq i \leq n-1$.
- $x_n(\cdot)$ is κ-Lipschitz over $[0,T]$ with $\|\dot{x}_n(t)\| \leq \kappa$ a.e. $t \in [0,T]$.
- $x_n \to x$ uniformly in $C([0,T];H)$, $x \in W^{1,\infty}([0,T];H)$ and
$$x(t) \in C(t) \text{ for all } t \in [0,T].$$
- $\dot{x}_n \to \dot{x}$ in $L^1([0,T];H)$.
- x is the unique solution of the Sweeping Process.

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Variants and extensions of Moreau's sweeping Process

- Perturbed state-dependent sweeping processes:

$$\begin{cases} -\dot{x}(t) \in N\left(C(t,x(t));x(t)\right) + F(t,x(t)) & \text{a.e. } t \in [0,T]; \\ x(0) = x_0 \in C(0), \end{cases}$$

- BV sweeping processes

$$\begin{cases} -dx \in N\left(C(t);x(t)\right) \\ x(0) = x_0 \in C(0), \end{cases}$$

- Second-order sweeping processes:

$$\begin{cases} -\ddot{x}(t) \in N\left(C(t,x(t),\dot{x}(t));\dot{x}(t)\right) + F(t,x(t),\dot{x}(t)) & \text{a.e. } t \in [0,T]; \\ x(0) = x_0, \dot{x}(0) = v_0 \in C(0,x_0,v_0) \end{cases}$$

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Perturbed sweeping processes

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) + F(t, x(t)) & \text{a.e. } t \in [0, T]; \\ x(0) = x_0 \in C(0), \end{cases}$$

where

- $(C(t))_{t\in[0,T]}$ is a family of nonempty closed sets.
- $N(S, \cdot)$ is some appropriate outward normal cone to S.
- $F\colon [0, T] \times H \rightrightarrows H$ is a set-valued map with nonempty closed convex values satisfying some standard conditions, e.g., upper semicontinuity and measurability.

Perturbed Sweeping Processes: Existence theory

Main contributions:

- J.-J. Moreau (1971, 1972, 1977, 1999)
  $\triangleright$ $C(t)$ convex and $F \equiv 0$.
- C. Castaing - T.D. Ha - M. Valadier (1993)
  $\triangleright$ $C(t)$ convex and complement of a convex and F usc.
- M. Kunze - Monteiro-Marques (1996, 2000)
  $\triangleright$ $C(t)$ convex and $F \equiv 0$.
- H. Benabdellah (1999)
  $\triangleright C(t)$ closed, $F \equiv 0$ and $\dim H < +\infty$.
- G. Colombo - V. Goncharov (1999)
  $\triangleright$ $C(t)$ closed, $F \equiv 0$ and $\dim H < +\infty$.
- M. Bounkhel - L. Thibault (2005) & Bounkhel (2012)
  $\triangleright$ $C(t)$ prox-regular and F usc.

The Perturbed Sweeping Process: Existence theory

Main contributions:

- J. Edmond - L. Thibault (2005, 2006)
  ▷ $C(t)$ prox-regular and F usc.
- T. Haddad - A. Jourani - L. Thibault (2008)
  ▷ $C(t)$ α-far, F mixed usc and $\dim H < +\infty$.
- L. Thibault (2003, 2008, 2016)
  ▷ $C(t)$ convex and prox-regular.
- S. Adly, L. Thibault, F. Nacry (2017)
  ▷ $C(t)$ prox-regular and f Lipschitz.
- A. Jourani - E. Vilches (2016, 2017)
  ▷ $C(t)$ α-far / subsmooth and F usc.
- V. Recupero (2015, 2016, 2018, 2020)
  ▷ BV case with $F \equiv 0$.

Moreau's perturbed sweeping process

Theorem (Jourani & Vilches, 2017 [4])

Assume that the following assumptions hold true:

- *F is a Carathéodory set-valued map.*
- *For some $\kappa \geq 0$*
$$\mathrm{Haus}(C(t), C(s)) \leq \kappa |t - s|.$$
- *The family $(C(t))_{t\in[0,T]}$ is positively α-far.*

Then, there exists at least one solution of

$$\begin{cases} -\dot{x}(t) \in N\left(C(t); x(t)\right) + F(t, x(t)) & a.e.\ t \in [0, T]; \\ x(0) = x_0 \in C(0). \end{cases}$$

[4] A. Jourani, E. Vilches, Galerkin-like method and generalized perturbed sweeping process with nonregular sets, *SIAM J. Control Optim.*, 55(4):2412-2436, 2017.

Perturbed state-dependent sweeping processes

$$\begin{cases} -\dot{x}(t) \in N\left(C(t,x(t)); x(t)\right) + F(t,x(t)) & \text{a.e. } t \in [0,T]; \\ x(0) = x_0 \in C(0), \end{cases}$$

where

- $(C(t,x))_{t\in[0,T],x\in H}$ is a family of nonempty closed sets.
- $N(S,\cdot)$ is some appropriate outward normal cone to S.
- $F\colon [0,T] \times H \rightrightarrows H$ is a set-valued map with nonempty closed convex values satisfying some standard conditions, e.g., upper semicontinuity and measurability.

State-dependent sweeping process

Theorem (Jourani & Vilches, 2017 [4])

Assume that the following assumptions hold true:

- *F is a Carathéodory set-valued map.*
- *For some $\kappa \geq 0$ and $L \in [0,1[$*

$$\mathrm{Haus}(C(t,x), C(s,y)) \leq \kappa|t-s| + L\|x-y\|.$$

- *The family $\{C(t,x) \colon (t,x) \in [0,T] \times H\}$ is equi-uniformly subsmooth.*

Then, there exists at least one solution of

$$\begin{cases} -\dot{x}(t) \in N\left(C(t,x(t)); x(t)\right) + F(t,x(t)) & a.e.\ t \in [0,T]; \\ x(0) = x_0 \in C(0,x_0). \end{cases}$$

[4] A. Jourani, E. Vilches, Galerkin-like method and generalized perturbed sweeping process with nonregular sets, *SIAM J. Control Optim.*, 55(4):2412-2436, 2017.

Second-order sweeping processes

$$\begin{cases} -\ddot{x}(t) \in N\left(C(t,x(t),\dot{x}(t));\dot{x}(t)\right) + F(t,x(t)) & \text{a.e. } t \in [0,T]; \\ x(0) = x_0 \in C(0), \end{cases}$$

where

- $(C(t,x,y))_{t\in[0,T],x,y\in H}$ is a family of nonempty closed sets.
- $N(S,\cdot)$ is some appropriate outward normal cone to S.
- $F\colon [0,T]\times H \rightrightarrows H$ is a set-valued map with nonempty closed convex values satisfying some standard conditions, e.g., upper semicontinuity and measurability.

Second-order sweeping process

Theorem (Jourani & Vilches, 2017 [4])

Assume that the following assumptions hold true:

- *F is a Carathéodory set-valued map.*
- *For some $\kappa \geq 0$, $L_1 \in [0,1[$ and $L_2 \geq 0$*

$$\operatorname{Haus}(C(t,x,z), C(s,y,w)) \leq \kappa|t-s| + L_1\|x-y\| + L_2\|z-w\|.$$

- *The family $\{C(t,x,y)\colon (t,x,y) \in [0,T] \times H \times H\}$ is equi-uniformly subsmooth.*

Then, there exists at least one solution of

$$\begin{cases} -\ddot{x}(t) \in N\left(C(t,x(t),\dot{x}(t)); \dot{x}(t)\right) + F(t,x(t)) & \text{a.e. } t \in [0,T]; \\ x(0) = x_0, \dot{x}(0) = v_0 \in C(0,x_0,v_0). \end{cases}$$

[4] A. Jourani, E. Vilches, Galerkin-like method and generalized perturbed sweeping process with nonregular sets, *SIAM J. Control Optim.*, 55(4):2412-2436, 2017.

Sweeping processes with nonlocal initial conditions

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); x(t)\right) + F(t, x(t)) & \text{a.e. } t \in [0, T], \\ x(0) = Mx \in C(0), \end{cases}$$

where $M\colon C\left([0,T]; H\right) \to H$ is a nonlocal operator, e.g.,

- $Mx = \pm x(T)$ (periodic and anti-periodic initial conditions);
- $Mx = \frac{1}{T-0}\int_0^T x(s)ds$ (mean value initial conditions);
- $Mx = \sum_{k=1}^{k_0} \alpha_i x(t_i)$ with $\alpha_i \in \mathbb{R}$ and $\sum_{i=1}^{k_0} |\alpha_i| \leq 1$, where $0 < t_1 < \cdots < t_{k_0} \leq T$ (multi-point initial condition).

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Implicit sweeping processes

Let us consider the so-called Implicit Sweeping Process

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); A\dot{x}(t) + Bx(t)\right) & \text{a.e. } t \in [0,T], \\ x(0) = x_0 \in H, \end{cases}$$

where

1. For all $t \in [0,T]$, the set $C(t)$ is a nonempty, closed and convex set.
2. $A, B\colon H \to H$ are bounded, linear and symmetric operators.

This variant was recently introduced by Adly et al ([1, 2]) and then extended by Migórski-Sofonea-Zeng [7] to history-dependent operators.

Quasistatic Variational Inequalities

Assume that the following conditions are satisfied:

- $K \subseteq H$ is a nonempty, closed, and convex cone.
- $a(\cdot,\cdot), b(\cdot,\cdot)\colon H \times H \to \mathbb{R}$ are two real continuous bilinear and symmetric forms such that for all $u \in H$ the condition $a(u,u) \geq \alpha\|u\|^2$ holds, for some $\alpha > 0$.
- $j\colon K \to \mathbb{R}$ is a convex function, positively homogeneous of degree 1 (i.e., $j(\lambda x) = \lambda j(x)$ for all $\lambda > 0$) and Lipschitz continuous with $j(0) = 0$.
- The function $f\colon [0,T] \to H$ belongs to $L^1([0,T];H)$.

Quasistatic Variational Inequalities

Let us consider the following evolution variational inequality:
Find $x\colon [0,T] \to H$ such that $\dot{x}(t) \in K$ a.e. $t \in [0,T]$ and for all a.e. $t \in [0,T]$ and all $y \in K$

$$\begin{aligned} &a(\dot{x}(t), y - \dot{x}(t)) + b(x(t), y - \dot{x}(t), y - \dot{x}(t)) \\ &+ j(y) - j(\dot{x}(t)) \geq \langle f(t), y - \dot{x}(t) \rangle . \end{aligned} \tag{4.1}$$

Thus, the problem (4.1) is equivalent to find $x\colon [0,T] \to H$ such that

$$f(t) - A\dot{x}(t) - Bx(t) \in \partial J(\dot{x}(t)) \quad \text{a.e. } t \in [0,T],$$

where $J(x) = j(x)$ for $x \in K$ and $J(x) = +\infty$ for $x \notin K$ and A and B are the linear operator associated with a and b, that is,

$$\begin{aligned} a(x,y) &= \langle Ax, y \rangle \\ b(x,y) &= \langle Bx, y \rangle \end{aligned}$$

Quasistatic Variational Inequalities

Finally the problem: Find $x\colon [0,T] \to H$ such that

$$f(t) - A\dot{x}(t) + Bx(t) \in \partial J(\dot{x}(t)) \quad \text{a.e. } t \in [0,T],$$

is equivalent to: Find $x\colon [0,T] \to H$ such that

$$\dot{x}(t) \in -N\left(C(t); A\dot{x}(t) + Bx(t)\right) \quad \text{a.e. } t \in [0,T],$$

where $C(t) = f(t) - \partial J(0)$.

Implicit Sweeping Processes

Thus, we are interested in the so-called Implicit Sweeping Process

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); A\dot{x}(t) + Bx(t)\right) & \text{a.e. } t \in [0,T], \\ x(0) = x_0 \in H, \end{cases}$$

where

1. For all $t \in [0,T]$, the set $C(t)$ is a nonempty, closed and convex set.
2. $A, B\colon H \to H$ is a bounded, linear and symmetric operator

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Implicit vs Moreau's Sweeping Processes

1. Implicit Sweeping Process

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); A\dot{x}(t) + Bx(t)\right) & \text{a.e. } t \in [0,T], \\ x(0) = x_0 \in H, \end{cases}$$

2. Moreau's Sweeping Process

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); x(t)\right) & \text{a.e. } t \in [0,T], \\ x(0) = x_0 \in H, \end{cases}$$

A Differential Equation Approach

The Implicit Sweeping Process is equivalent to the following differential equation:

$$\begin{cases} \dot{x}(t) = -P^{-1}QBx(t) + P^{-1}\operatorname{proj}_{QC(t)}(QBx(t)) & \text{a.e. } t \in [0,T], \\ x(0) = x_0 \in H, \end{cases} \tag{4.2}$$

where $A = P^2$ and $Q := (P)^{-1}$.

Important remark:

If the set-valued map $C\colon [0,T] \rightrightarrows H$ is measurable and the function $t \mapsto d_{C(t)}(0)$ is integrable, then the map

$$(t,x) \to -P^{-1}QBx + P^{-1}\operatorname{proj}_{QC(t)}(QBx)$$

satisfies the hypotheses of the Cauchy-Lipschitz-Picard Theorem.

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An existence result

Theorem (Jourani-Vilches, 2019 [5])

Assume that the following conditions hold:

- $A, B\colon H \to H$ *are linear, bounded and symmetric operators such that* $A = P^2$ *for some invertible operator* P.
- *The set-valued map* $C\colon [0,T] \rightrightarrows H$ *is measurable and the function* $t \mapsto d_{C(t)}(0)$ *is integrable.*

Then for any initial point $x_0 \in H$ *there exists a unique absolutely continuous mapping* $x(\cdot; x_0)\colon [0,T] \to H$ *satisfying*

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); A\dot{x}(t) + Bx(t)\right) \ \ a.e.\ t \in [0,T], \\ x(0) = x_0. \end{cases}$$

Moreover, the map $x_0 \mapsto x(\cdot; x_0)$ *is locally Lipschitz continuous.*

[5] A. Jourani, E. Vilches, A differential equation approach to implicit sweeping processes, *J. Differential Equations*, 266(8):5168-5184, 2019.

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History-dependent operators

Definition

An operator $\mathcal{R}\colon C([0,T];H) \to C([0,T];H)$ is called History-Dependent if there exists a constant $L_{\mathcal{R}} \geq 0$ such that

$$\|(\mathcal{R}x)(t) - (\mathcal{R}y)(t)\| \leq L_{\mathcal{R}} \int_0^t \|x(s) - y(s)\|\, ds$$

for all $x, y \in C([0,T];H)$ and $t \in [0,T]$.

Such operators arise in Solid Mechanics and Contact Mechanics.

History-dependent operators

An important property of history-dependent operators is provided by the following fixed point principle

Lemma ([10])

If $\mathcal{R}\colon C([0,T];H) \to C([0,T];H)$ is a history-dependent operator, then there exists a unique function $x^* \in C([0,T];H)$ such that

$$\mathcal{R}x^* = x^*.$$

[10] Sofonea, M., Avramescu, C. and Matei A., A fixed point result with applications in the study of viscoplastic frictionless contact problems, *Commun. Pure Appl. Anal.*, 7(8):645-658, 2008.

History-dependent implicit sweeping processes

Let us consider the differential inclusion: find $x\colon [0,T] \to H$ such that

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}\left(A\dot{x}(t) + Bx(t) + (\mathcal{R}x)(t)\right) & \text{a.e. } t \in [0,T], \\ x(0) = x_0, \end{cases} \tag{5.1}$$

where

- for all $t \in [0,T]$, the set $C(t)$ is a nonempty, closed and convex set,
- $A\colon H \to H$ is a bounded linear operator,
- $B\colon H \to H$ is a Lipschitz operator,
- $\mathcal{R}$ is a history-dependent operator.

An existence result

Theorem (Vilches - Zeng, 2020 [11])

Assume that the following conditions hold:

- $A, B\colon H \to H$ *are linear, bounded and symmetric operators such that* $A = P^2$ *for some invertible operator* P.
- *The set-valued map* $C\colon [0,T] \rightrightarrows H$ *is measurable and the function* $t \mapsto d_{C(t)}(0)$ *is integrable.*
- $\mathcal{R}$ *is a history-dependent operator.*

Then for any initial point $x_0 \in H$ *there exists a unique absolutely continuous mapping* $x(\cdot; x_0)\colon [0,T] \to H$ *satisfying* $x(0) = x_0$ *and*

$$\dot{x}(t) \in -N\left(C(t); A\dot{x}(t) + Bx(t) + (\mathcal{R}x)(t)\right) \; a.e. \; t \in [0,T].$$

Moreover, the map $x_0 \mapsto x(\cdot; x_0)$ *is locally Lipschitz continuous.*

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[11] E.Vilches, S. Zeng, Well-Posedness of History/State-Dependent Implicit Sweeping Processes, *J. Optim. Theory Appl.*, 2020.

Sketch of proof

Given $w \in C([0,T];H)$, let us consider the unique solution $x\colon [0,T] \to H$ of the problem

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); A\dot{x}(t) + Bx(t) + (\mathcal{R}w)(t)\right) \text{ a.e. } t \in [0,T], \\ x(0) = x_0. \end{cases}$$

We introduce the operator $\mathcal{S}\colon C([0,T];H) \to C([0,T];H)$ defined by

$$\mathcal{S}w = x(w).$$

It is clear that any fixed point of $\mathcal{S}$ is a solution for the History-dependent sweeping process.

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Sketch of proof

Proposition

The operator $\mathcal{S}$ is a history-dependent operator, that is, there exists a constant $L_{\mathcal{S}} \geq 0$ such that

$$\|(\mathcal{S}x)(t) - (\mathcal{S}y)(t)\| \leq L_{\mathcal{S}} \int_0^t \|x(s) - y(s)\|\, ds$$

for all $x, y \in C\left([0,T];H\right)$ and $t \in [0,T]$.

Finally, it follows from the fixed point property for history-dependent operators that $\mathcal{S}$ has a fixed point.

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Moreau's Sweeping Processes and its variants

1. Perturbed Sweeping Processes
2. Degenerate Sweeping Processes
3. State-dependent Sweeping Processes
4. Second-order Sweeping Processes
5. Implicit Sweeping Processes
6. History-dependent Implicit Sweeping Processes
7. etc.

Future work and open problems

1. Existence of solutions.
2. Stability and asymptotic behavior of sweeping processes and its variants.
3. Optimal control of sweeping processes.
4. Applications in nonsmooth mechanics.
5. Applications in electrical circuits.

We refer to the the following survey for a detailed overview on the subject:

[3] Brogliato, B. and Tanwani, A., Dynamical Systems Coupled with Monotone Set-Valued Operators: Formalisms, Applications, Well-Posedness, and Stability. *SIAM Rev.*, 62(1):3-129, 2020.

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An Overview of Moreau's Sweeping Processes with Applications

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