Principal-Agent model in insurance: from the static to the continuous-time setting

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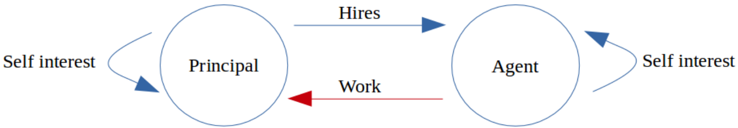
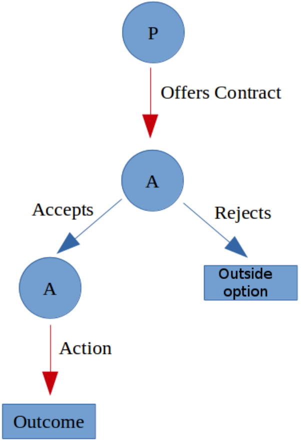
The P-A model

The **Principal**(she) hires the **Agent**(he) to work on her behalf.



The P-A model

The interaction between both sides is **sequential**.



From the game theory point of view, the **Principal** and the **Agent** play a Non-zero sum **Stackelberg game**

The P-A model

Three main types of **Principal**-**Agent** problems are studied in the literature:

1. First-best / risk sharing (**Observable** actions).

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3. Third-best / adverse selection (**Hidden** types).

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1. First-best / risk sharing (**Observable** actions). $\rightarrow$ optimization problem.
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3. Third-best / adverse selection (**Hidden** types). $\rightarrow$ mechanism design problem.

Role of time

Static model: Mirrlees (1975), Hölmstrom (1979), Rogerson (1985).

Output $\in \mathbb{R} \Longrightarrow$ Optimization with **variational inequality** constrain.

Discrete-time model: Spear and Srivastava (1987), Chiappori, Macho, Rey and Salanié (1994).

Output $\in \mathbb{R}^n \Longrightarrow$ **Dynamic programming** optimization.

Continuous-time model: Hoölmstrom and Milgrom (1987), Sannikov (2007), Cvitanić, Possamaï and Touzi (2017).

The output is a stochastic process $\Longrightarrow$ **Stochastic control** problem.

P-A examples

Examples of **Principal**-**Agent** situations:

- Agency problems: **Shareholders**-**Manager**.
- Bank monitoring: **Investor**-**Bank**.
- Electricity pricing: **Provider**-**Consumers**.
- Pollution: **Regulator**-**Companies**.
- Insurance: **Insurer**-**Insured person**.

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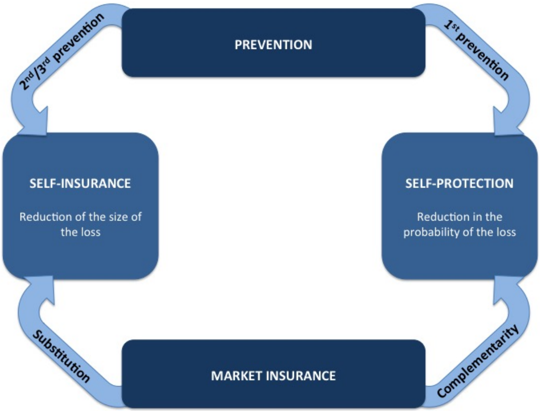
1 Principal-Agent model

2 Static model: self-insurance and self-protection
- Solving the problem of IB
- Solving the problem of IS

3 Continuous-time model: self-protection

Motivation

Ehrlich and Becker (1972), Courbage, Rey and Treich (2013).



Model and Assumptions

- S. Bensalem, N. Hernández, N. Kazi-Tani. Prevention efforts, insurance demand and price incentives under coherent risk measures, 2020. Insurance: Mathematics and Economics, 93, 369-386.

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Model: On $(\Omega, \mathcal{F}, \mathbb{P})$ a fixed probability space.
▶ We consider a family of non-negative r.v. $(X_e)_{e\in[0,+\infty)}$ representing the losses, of distributions μ_e **decreasing** for the first order stochastic dominance.

$$e_1 < e_2 \Longrightarrow \mathbb{E}[f(X_{e_1})] > \mathbb{E}[f(X_{e_2})], \qquad \forall f \text{ non-decreasing}$$

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▶ We consider a law-invariant **coherent** risk measure ρ

ρ is **monotone**: $X \leq Y$ almost surely $\Rightarrow \rho(X) \leq \rho(Y)$.

ρ is **cash-additive**: $\forall\ m \in \mathbb{R},\ \rho(X+m) = \rho(X) + m$.

ρ is **convex**: for $X, Y \in L^1$ and $\forall\lambda \in [0,1]$,
$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$.

ρ is **positively homogeneous**: $\forall\lambda \in \mathbb{R}^+,\ \rho(\lambda X) = \lambda\rho(X)$.

ρ is **law-invariant**: $X \overset{d}{=} Y$ implies that $\rho(X) = \rho(Y)$.

Example of law-invariant coherent risk measure

Example: **Distortion** *risk measures defined by*

$$\rho(X_e) := \int_0^1 \bar{q}_{X_e}(u)\psi(du),$$

where

- $\bar{q}_{X_e}$ *is the* **tail quantile** *function (non-increasing).*
- $\psi : [0,1] \to [0,1]$ *is a* **concave distortion** *function.*

IB characteristics:

- risk with loss X_e.
- proportional insurance contract with insurance level $\alpha \in [0,1]$.
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Goal: minimize his total loss

$$\inf_{(\alpha,e)\in[0,1]\times(0,\infty)} \{ \underbrace{(1-\alpha)\rho_1(X_e)}_{\text{uninsured loss}} + \underbrace{\Pi(\alpha X_e)}_{\text{insurance premium}} + \underbrace{c(e)}_{\text{cost of effort}} \}. \quad (1)$$

where ρ_1 is the coherent risk measure used by IB.

- Given a premium Π, the solution to (1) is denoted by $(\alpha^*(\Pi), e^*(\Pi))$.

The optimization problem

IS characteristics:

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where ρ_2 is the coherent risk measure used by IS.

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Equivalently:

$$\inf_{(\theta,\alpha,e)\in[0,\infty)\times[0,1]\times[0,\infty)} \{\alpha\rho_2(X_e) - \alpha(1+\theta)\mathbb{E}[X_e]\},$$

subject to

$$(\alpha, e) \in \operatorname*{argmin}_{(\alpha',e')\in[0,1]\times(0,\infty)} \{(1-\alpha')\rho_1(X_{e'}) + \alpha'(1+\theta)\mathbb{E}[X_{e'}] + c(e')\}.$$

Optimal insurance coverage

The optimal insurance coverage is bang-bang.

Indeed

$$\alpha^\star(e) = \begin{cases} 1, & \text{if } e \in \mathcal{I}, \\ 0, & \text{if } e \in \mathcal{N} := \mathcal{I}^c, \end{cases}$$

where $\mathcal{I} := \{ e \in \mathbb{R}^+ | \; \Pi(X_e) \leq \rho_1(X_e) \}$.

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where $\mathcal{I} := \left\{ e \in \mathbb{R}^+ \middle| \, \Pi(X_e) \leq \rho_1(X_e) \right\}$.

- Since $\Pi(X) = (1+\theta)\mathbb{E}[X]$, we have

$$\mathcal{I} := \left\{ e \in \mathbb{R}^+ \middle| \, 1+\theta \leq G(e) \right\},$$
$$\mathcal{N} := \left\{ e \in \mathbb{R}^+ \middle| \, 1+\theta > G(e) \right\},$$

where $G(e) := \frac{\rho_1(X_e)}{\mathbb{E}[X_e]}$. We assume that G is **monotonic**, depending on how the effort impacts the risk and the price of insurance.

Case 1: G **non-increasing**: an increased effort has a bigger impact on the risk (**self-insurance** models).

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Let

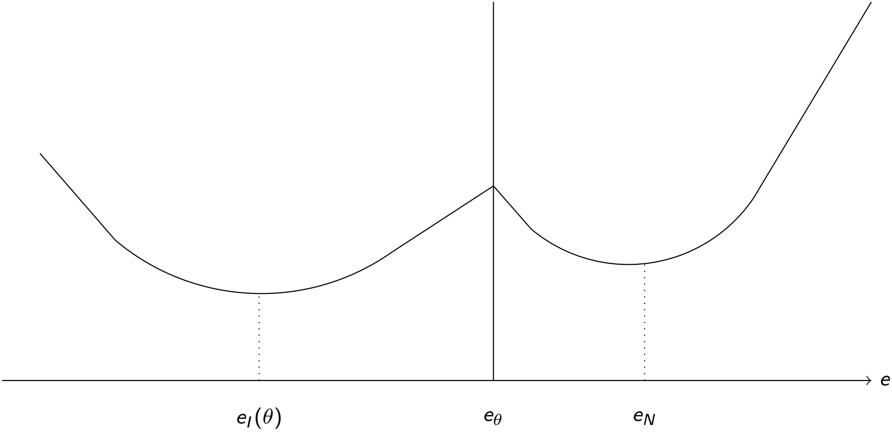
$$L_1(\alpha^\star(e), e) = (1-\alpha)\rho_1(X_e) + \alpha(1+\theta)\mathbb{E}[X_e] + c(e)$$

Then

$$L_1(\alpha^\star(e), e) = \begin{cases} \rho_1(X_e) + c(e) =: L_{\mathcal{N}}(e), & \text{if } e \in \mathcal{N}, \\ (1+\theta)\mathbb{E}[X_e] + c(e) =: L^\theta_{\mathcal{I}}(e), & \text{if } e \in \mathcal{I}. \end{cases}$$

Optimal level of self-insurance

For θ fixed we define the local minima of L_1 respectively on $\mathcal{I}$ and $\mathcal{N}$:



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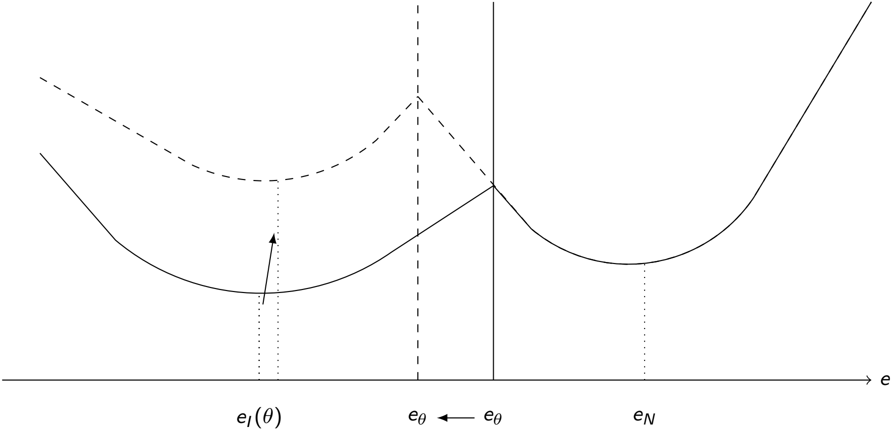

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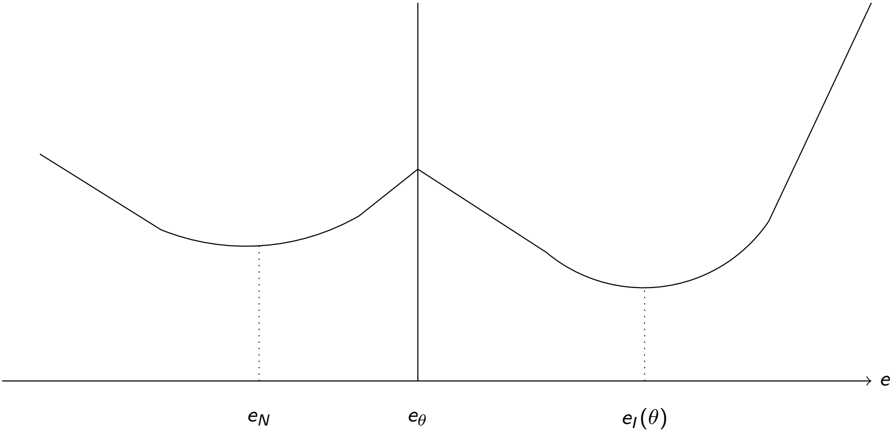
- Same resolution as previously.

Optimal level of self-protection



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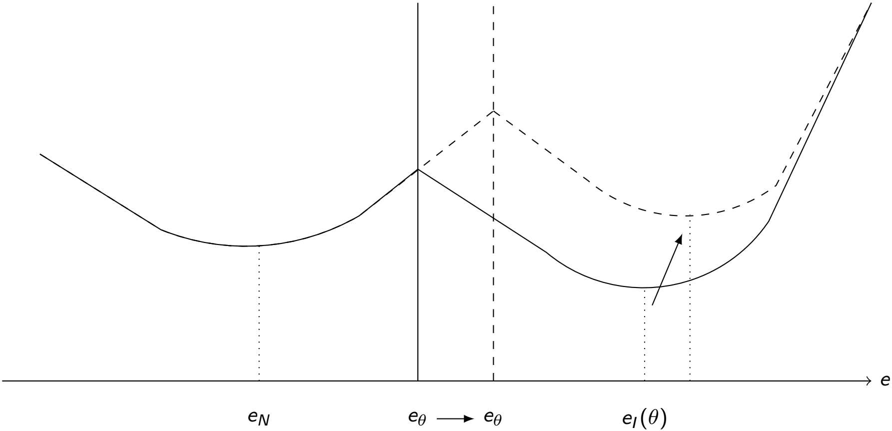

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Optimal insurance price

Problem of the seller:

- IS wants to find the best premium for the insurance coverage that IB is willing to accept

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- The problem of IS is then reduced to a **minimization over a compact set**

$$\min_{[0,\theta_M]} \rho_2(X_{e_I(\theta)}) - (1+\theta)\mathbb{E}[X_{e_I(\theta)}].$$

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$$\min_{[0,\theta_M]} \rho_2(X_{e_I(\theta)}) - (1+\theta)\mathbb{E}[X_{e_I(\theta)}].$$

- By **continuity** there exists an optimal loading factor $\theta^\star$ which minimizes the risk measure of the loss of IS.

Example:

$$P_{X_e} = (1-p)\delta_0 + pP_{Y_e},$$

where $0 < p < 1$, δ_0 *is the Dirac mass at 0 and* P_{Y_e} *denotes the distribution of a Pareto random variable of parameters* $\hat{x} > 0$ *and* $k > 0$*.*

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IS's problem is easier to solve because the objective function is **monotone** *on a compact set. The solution of the problem is given by* $\theta^\star = \theta_M$, *the* **maximum premium** *IB is willing to pay.*

Conclusion

- There are **two optimal levels for insurance coverage** for the insurance buyer: total insurance and no insurance.
- For each of these level, there is an **optimal level of self-insurance** and an **optimal level of self-protection** for the insurance buyer.
- There is an **optimal insurance price** on the supply side for which the buyer will subscribe a full coverage and will exert his optimal level of prevention.
- We can implement and run this model econometrically.
- Market insurance and self-insurance are **substitutes**.
- Market insurance and self-protection can be **complements**.

Each accident is modeled as the jump of a compound Poisson process J

$$J_t = \sum_{i=1}^{N_t} Z_i, \ 0 \leq t \leq T.$$

In our **self-protection** model, the agent can control the **intensity** of the Poisson process N, by doing continuous-time effort e_t. The wealth process of the agent follows the dynamics

$$X_t^e := \omega_0 - \Pi(\alpha) - (1-\alpha)J_t - \int_0^t c(e_s)ds, \ 0 \leq t \leq T. \tag{3}$$

The utility maximization problem of the insurance buyer is

$$V_A := \sup_{(\alpha,e)\in[0,1]\times\Sigma} \mathbb{E}^e\left[-\frac{1}{\gamma}e^{-\gamma\left(-\Pi(\alpha)-(1-\alpha)\int_0^T\int_0^\infty x\mu_J(ds,dx)-\int_0^T c(e_s)ds\right)}\right]$$

where V_A is the value function of the Agent. The problem can be divided in two problems, one in α and one in e so we can start by finding the optimal effort of self-protection. The problem becomes then

$$V_A = \sup_{\alpha\in[0,1]} e^{\gamma\Pi(\alpha)}V_A^\alpha,$$

where

$$V_A^\alpha := \sup_{e\in\Sigma}\mathbb{E}^e\left[-\frac{1}{\gamma}e^{-\gamma\left(-(1-\alpha)\int_0^T\int_0^\infty x\mu_J(ds,dx)-\int_0^T c(e_s)ds\right)}\right].$$

Theorem

For fixed $\alpha \in [0,1]$, the value function of the Agent is given by

$$V_A^\alpha = -\frac{1}{\gamma} e^{-\gamma Y_0^\alpha},$$

where (Y^α, H^α) is the unique solution to the BSDEJ

$$Y_t^\alpha = \int_t^T \int_0^\infty H_s^\alpha(x) \tilde{\mu}_J^0(ds, dx) - \int_t^T F^\alpha(H_s^\alpha) ds, \ 0 \le t \le T, \tag{4}$$

with

$$f^\alpha(e,h) := c(e) + \frac{\lambda(e)}{\gamma} \int_0^\infty \left(e^{\gamma(h+(1-\alpha)x)} - 1 \right) dG(x) - \int_0^\infty \lambda_0 h dG(x),$$

and

$$F^\alpha(h) := \inf_{e \in [0,\infty)} f^\alpha(e,h). \tag{5}$$

The optimal prevention effort for the agent is given by

$$e_t^\star = e^\star(H_t^\alpha),$$

where $e^\star(h)$ denotes the minimizer in (5).

Recall the problem

$$V_A = \sup_{\alpha \in [0,1]} e^{\gamma \Pi(\alpha)} V_A^{\alpha}.$$

Theorem

Assume the insurance premium Π is a lower semicontinuous, non-decreasing function of α. Then, there exists an optimal insurance cover, denoted $\alpha^\star$. The value function of the agent is then given by $V_A = V_A^{\alpha^\star} e^{\gamma \Pi(\alpha^\star)}$.

Example:

Suppose that $Z_i = 1$ for all i so then J is equal to the Poisson process N of intensity $\lambda(e)$ under $\mathbb{P}^e$.

$$\lambda(e) = \begin{cases} 1 - e & \text{if } e \leq 1 - \epsilon, \\ \epsilon & \text{if } e > 1 - \epsilon, \end{cases}$$

Exerting an effort e cost to the Agent the value $c(e) = ce$ with $c > 0$.

Proposition

The unique solution of the BSDE (4) is given by (Y^α, H^α) such that

$$\begin{cases} Y_t^\alpha = -\left(ce^\star + \frac{\lambda(e^\star)}{\gamma}(e^{\gamma(1-\alpha)} - 1)\right)(T-t), & 0 \leq t \leq T, \\ H_t^\alpha \equiv 0, & 0 \leq t \leq T, \end{cases}$$

where the optimal effort $e^\star \equiv 0$ if $c\gamma \geq e^{\gamma(1-\alpha)} - 1$ and $e^\star \equiv 1 - \epsilon$ otherwise.

Proposition

The value $V_A^\alpha = -\frac{1}{\gamma}e^{-\gamma Y_0^\alpha}$ is given by

(1) If $c\gamma \geq e^{\gamma(1-\alpha)} - 1$ then $V_A^\alpha = -\frac{1}{\gamma}e^{(e^{\gamma(1-\alpha)}-1)T}$.

(2) If $c\gamma < e^{\gamma(1-\alpha)} - 1$ then $V_A^\alpha = -\frac{1}{\gamma}e^{(\epsilon(e^{\gamma(1-\alpha)}-1)+\gamma c(1-\epsilon))T}$.

Now, assume that the insurance premium is given by

$$\Pi(\alpha) := \alpha(1+\theta)\mathbb{E}[N_T] = \alpha(1+\theta)T,$$

where $\theta \geq 0$ is the safety loading that reflects the insurance price.

Proposition

The optimal insurance cover $\alpha^\star$ has the following dependence on the safety load θ

(1) If $1+\theta \leq \lambda(e^\star)e^\gamma$, then $\alpha^\star = 1 - \frac{\log(1+\theta)-\log(\lambda(e^\star))}{\gamma}$.

(2) If $1+\theta > \lambda(e^\star)e^\gamma$, then $\alpha^\star = 0$.

- For small prices, there is **complementarity** between market insurance and $\alpha^\star$.

Thank you for your attention