σ-Convex functions and σ-Subdifferentials

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- Convex functions and monotone operators

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- Subdifferential and σ-Subdifferential

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We recall that a function f is called *quasi-convex* if for each $x, y \in X$ and for every $t \in [0, 1]$,

$$f((1-t)x + ty) \leq \max\{f(x), f(y)\}.$$

A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called *lower semicontinuous* (briefly, lsc) at $x_0 \in X$ if $f(x_0) \leq \liminf_{x \to x_0} f(x)$.

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If $f(x) \in \mathbb{R}$, then the *subdifferential* of f at x is denoted by $\partial f(x)$ and is defined as the set of all $x^* \in X^*$ satisfying

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For two multivalued operators T and S we write $T \subseteq S$ if S is an extension of T.

Definition

A set $M \subset X \times X^*$ is
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(ii) *maximal monotone* if it is monotone and it is not properly included in any other monotone subset of $X \times X^*$. That is, if M_1 is a monotone subset of $X \times X^*$ and $M \subset M_1$, then $M = M_1$.

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We say that an element $(x, x^*) \in X \times X^*$ is *monotonically related* to the M if $\langle y^* - x^*, y - x \rangle \geq 0$ for all $(y, y^*) \in M$.

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In the next definition, we will formulate the definition of monotone operators in terms of their graphs.

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The multifunction $J(\cdot) := \partial(\frac{1}{2}\|\cdot\|^2) : X \to 2^{X^*}$ is called the duality mapping of X.
The following holds

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Note that since $(\frac{1}{2}\|\cdot\|^2)$ is proper, lsc and convex, J is maximal monotone. When X is a Hilbert space, then $J = I$.

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Note that when $\sigma(x) = \epsilon$, (ϵ is a constant positive real number) the definition of σ-monotonicity reduces to ϵ-monotonicity

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$$(2) \quad f\left(tx + (1-t)\,y\right) \leq tf\left(x\right) + (1-t)\,f\left(y\right) + t\left(1-t\right)\min\{\sigma\left(x\right), \sigma\left(y\right)\}||x-y||$$

for all $x, y \in X$, and $t \in]0,1[$.

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There are σ-convex functions which are not ε-convex for any $\varepsilon \geq 0$, as shown in the following example.

Example

Consider the functions $\varphi, f, \sigma : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} x \sin^2 x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

$$\sigma(x) = \max \left\{ \varphi(x), \max_{z \leq x} \varphi(z) - \varphi(x) \right\}$$

$$f(x) = \int_0^x \varphi(t) dt.$$

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Note that if f is a σ-convex function, then $\operatorname{dom} f$ is a convex set.

Elementary properties

Proposition

(i) Suppose that f_1 *and* f_2 *are* σ_1*-convex and* σ_2*-convex, respectively, with* $\operatorname{dom} f_1 \cap \operatorname{dom} f_2 \neq \emptyset$ *and* $\alpha > 0$*. Then* $\alpha f_1 + f_2$ *is* $(\alpha\sigma_1 + \sigma_2)$*-convex.*

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(iv) Let $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a function. Then* f *is convex if and only if it is* σ*-convex for every* $\sigma : \operatorname{dom} f \to \mathbb{R}_+$*.*

Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, we define the map $\sigma_f : \operatorname{dom} f \to \mathbb{R}_+ \cup \{+\infty\}$ by

$$\sigma_f(x) = \inf\{a \in \mathbb{R}_+ : \frac{f(tx + (1-t)y) - tf(x) - (1-t)f(y)}{t(1-t)} \leq a\|x-y\|, \forall y \in \operatorname{dom} f, t \in]0,1[\}.$$

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It should be noticed that if f is σ'-convex for some $\sigma' : \operatorname{dom} f \to \mathbb{R}_+$, then

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In this case, σ_f is finite and f is σ_f-convex. Note that σ_f is the minimal σ such that f is σ-convex.

Explicit formula

Proposition

Suppose that f is σ-convex for some σ. Then

$$(5) \quad \sigma_f(x) = \max\left\{0, \sup_{t\in]0,1[}\ \sup_{y\in \operatorname{dom} f\setminus\{x\}} \frac{f(tx+(1-t)y) - tf(x) - (1-t)f(y)}{t(1-t)\|x-y\|}\right\}.$$

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Remark

Let $\{f_i\}_{i\in I}$ be an arbitrary family of σ-convex functions. If $f(x) = \sup_{i\in I} f_i(x)$, $x \in X$, then f is σ-convex.

Topological proprieties

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Lemma

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, σ-convex function. Assume that either X is finite-dimensional, or that f is lsc and X is a Banach space. Then f is locally bounded from above in the interior of its domain.

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Lemma

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a σ-convex function. Assume that f is bounded from above in a neighborhood of some point x_0. Then f is locally bounded from above in the interior of its domain.

Property B

We introduce the following assumption:

We say that the function σ has the property B, if for every $x \in \operatorname{int}\operatorname{dom} f$ and every $\varepsilon > 0$ sufficiently small, σ is bounded on the sphere $S(x, \varepsilon) = \{y \in X : \|x - y\| = \varepsilon\}$.

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in X . Note that this assumption is weaker than assuming that σ is locally bounded. For example, the function σ such that $\sigma(x) = 1/\|x\|$ for $x \neq 0$ and $\sigma(0) = 1$, satisfies property B without being locally bounded.

Theorem

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a* σ*-convex function. Assume that* f *is locally bounded from above in the interior of its domain. If* σ *satisfies property B, then* f *is locally Lipschitz in the interior of its domain.*

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Corollary

Every proper, σ*-convex function* $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *is locally Lipschitz in the interior of its domain.*

Clarke-Rockafellar Directional Derivative

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For a proper function $f : X \to \mathbb{R} \cup \{+\infty\}$ the Clarke-Rockafellar generalized directional derivative at x in a direction $z \in X$ is defined by

$$f^{\uparrow}(x,z) = \sup_{\delta>0} \limsup_{(y,\alpha)\xrightarrow{f}x,\lambda\searrow 0} \inf_{u\in B(z,\delta)} \frac{f(y+\lambda u)-\alpha}{\lambda}$$

where $(y,\alpha) \xrightarrow{f} x$ means that $y \to x, \alpha \to f(x)$ and $\alpha \geq f(y)$.

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If f is lsc at x, the above definition coincides with

$$f^{\uparrow}(x,z) = \sup_{\delta>0} \limsup_{y \xrightarrow{f} x, \lambda \searrow 0} \inf_{u \in B(z,\delta)} \frac{f(y+\lambda u) - f(y)}{\lambda}.$$

Here, $y \xrightarrow{f} x$ means that $y \to x$ and $f(y) \to f(x)$.

Clarke-Rockafellar Subdifferential

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The Clarke-Rockafellar subdifferential of f at $x \in \operatorname{dom} f$ is defined by

$$\partial^{CR} f(x) = \left\{ x^* \in X^* : \langle x^*, z \rangle \leq f^{\uparrow}(x, z) \quad \forall z \in X \right\}.$$

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In the following we introduce the notion of σ-subdifferential.

σ-Subdifferential

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Definition

Suppose that $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper function. The σ-subdifferential of f is the multivalued operator $\partial^\sigma f : X \to 2^{X^*}$ defined by

$$\partial^\sigma f(x) := \{x^* : \langle x^*, z\rangle \leq f(x+z) - f(x) + \min\{\sigma(x), \sigma(z+x)\} \|z\| \ \ \forall z \in X\}$$

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Equivalently,

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It follows from the above definition that $\partial f \subset \partial^\sigma f$ and so $D(\partial f) \subset D(\partial^\sigma f) \subset \operatorname{dom} f$. In the next proposition, we find a relationship between $\partial^{CR} f(x)$ and $\partial^\sigma f(x)$.

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Note that the function $f(x) = -|x|$ is σ-convex with $\sigma \equiv 2$. Then $\partial f(0) = \emptyset$, and $\partial^{CR}(f(0)) = [-1, 1]$ also it is easy to see that $\partial^{\sigma} f(0) = [-1, 1]$. On the other hand, if we take $\sigma' \equiv 4$, then f is σ'-convex and $\partial^{\sigma'}(0) = [-3, 3]$. Thus the inclusion in the above proposition can be equality or strict.

We note that if $f, g : X \to \mathbb{R} \cup \{+\infty\}$ are proper functions, then for each $x \in \operatorname{dom} f \cap \operatorname{dom} g$

$$\partial^{\sigma} f(x) + \partial^{\sigma} g(x) \subset \partial^{2\sigma}(f+g)(x). \tag{6}$$

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