An algebraic view of the smallest strictly monotonic function

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The concepts of solution of several optimization problems whose image set is a real topological linear space can be defined in the following way:

$\bar{x}$ is a solution if $M(\bar{x}) \cap (-E) = \emptyset$

X is an arbitrary decision set

Y real topological linear space

$M : X \rightrightarrows Y$

$E \subset Y$ satisfies certain algebraic conditions (convex, free-disposal, coradiant,...)

Examples: vector optimization problems, vector equilibrium problems, vector variational inequality problems, vector complementarity problems

$\bar{x}$ is a solution if $M(\bar{x}) \cap (-E) = \emptyset$

Example: vector optimization problems

$$\text{Min}_K\{f(x) : x \in S\} \tag{VP}$$

$f : X \to Y, \emptyset \neq S \subset X$

$\emptyset \neq K \subset Y$ convex cone

The decision maker's preferences are defined by the partial order $\leq_K$:

$$y_1, y_2 \in Y, \quad y_1 \leq_K y_2 \iff y_1 - y_2 \in -K$$

A point $\bar{x} \in S$ is an efficient (nondominated) solution of (VP) if

$$x \in S, f(x) \leq_K f(\bar{x}) \Rightarrow f(x) = f(\bar{x})$$

Equivalently,

$$(f(S) - f(\bar{x})) \cap (-K \setminus \{0\}) = \emptyset$$

$\bar{x}$ is a solution if $M(\bar{x}) \cap (-E) = \emptyset$

Convex separation and linear scalarization: if $M(\bar{x})$ and E are convex and a separation theorem can be applied, then there exists $\lambda \in Y' \setminus \{0\}$ such that

$$\inf_{y \in M(\bar{x})} \lambda(y) \geq \sup_{e \in E} \lambda(-e) \quad \longrightarrow \quad \text{“scalar optimization”}$$

Nonconvex separation and nonlinear scalarization: If $\varphi : Y \to \mathbb{R} \cup \{\pm\infty\}$ satisfies

$$\{y \in Y : \varphi(y) < c\} = -E$$

then

$$\inf_{y \in M(\bar{x})} \varphi(y) \geq c \quad \longrightarrow \quad \text{“scalar optimization”}$$

$\bar{x}$ is a solution if $M(\bar{x}) \cap (-E) = \emptyset$

Nonconvex separation and nonlinear scalarization: If $\varphi : Y \to \mathbb{R} \cup \{\pm\infty\}$ satisfies

$$\{y \in Y : \varphi(y) < c\} = -E$$

$$\inf_{y \in M(\bar{x})} \varphi(y) \geq c \quad \longrightarrow \quad \text{“scalar optimization”}$$

Nonconvex separation functionals

If Y is a real locally convex Hausdorff topological linear space and $q \in Y \backslash \{0\}$, then we can consider the so-called smallest strictly monotonic functional $\varphi_E^q : Y \to \mathbb{R} \cup \{\pm\infty\}$ (Gerth-Weidner (1990), Luc (1989), Luenberger (1992), Pascoletti-Serafini (1984), Rubinov (1977),. . .)

$$\varphi_E^q(y) := \inf\{t \in \mathbb{R} : y \in tq - E\} \quad \forall y \in Y$$

If Y is a normed space, then we can consider the oriented distance $\Delta_{-E} : Y \to \mathbb{R}$ (Hiriart-Urruty (1979))

$$\Delta_{-E}(y) := d(y, -E) - d(y, Y \backslash (-E)) \quad \forall y \in Y$$

Basic properties (Göpfert et al. (2003))

$$D + [0, \infty) \cdot k^0 \subset D, \qquad (2.22)$$

$$\varphi(y) := \inf\{t \in \mathbb{R} \mid (y,t) \in D'\} = \inf\{t \in \mathbb{R} \mid y \in tk^0 - D\}. \qquad (2.23)$$

Theorem 2.3.1. *Let $D \subset Y$ be a closed proper set and $k^0 \in Y$ be such that (2.22) holds. Then φ is l.s.c., $\operatorname{dom}\varphi = \mathbb{R}k^0 - D$,*

$$\{y \in Y \mid \varphi(y) \le \lambda\} = \lambda k^0 - D \quad \forall\ \lambda \in \mathbb{R}, \qquad (2.24)$$

and

$$\varphi(y + \lambda k^0) = \varphi(y) + \lambda \quad \forall\, y \in Y,\ \forall\, \lambda \in \mathbb{R}. \qquad (2.25)$$

Basic properties (Göpfert et al. (2003))

Suppose, furthermore, that

$$D + (0, \infty) \cdot k^0 \subset \operatorname{int} D. \tag{2.28}$$

Then

(f) φ *is continuous and*

$$\{y \in Y \mid \varphi(y) < \lambda\} = \lambda k^0 - \operatorname{int} D, \quad \forall \ \lambda \in \mathbb{R}, \tag{2.29}$$

$$\{y \in Y \mid \varphi(y) = \lambda\} = \lambda k^0 - \operatorname{bd} D, \quad \forall \ \lambda \in \mathbb{R}. \tag{2.30}$$

(g) *If* φ *is proper, then* φ *is* B*-monotone* $\Leftrightarrow D + B \subset D \Leftrightarrow \operatorname{bd} D + B \subset D$. *Moreover, if* φ *is finite-valued, then* φ ***strictly*** B***-monotone*** *(i.e.,* $y_2 - y_1 \in B \setminus \{0\} \Rightarrow \varphi(y_1) < \varphi(y_2)$*)* $\Leftrightarrow D + (B \setminus \{0\}) \subset \operatorname{int} D \Leftrightarrow \operatorname{bd} D + (B \setminus \{0\}) \subset \operatorname{int} D$.

Basic properties (Flores-Bazán et al. (2015))

([9, 10]), is the function $\xi_{q,S} : L \longrightarrow I\!R \cup \{\pm\infty\}$ defined by

$$\xi_{q,S}(y) := \inf\{t \in I\!R : y \in tq + S\}, \quad (2)$$

Proposition 4.1 *Let $\lambda \in I\!R$, $0 \neq q \in L$, and $\emptyset \neq A \subsetneq L$. The following assertions hold:*

(a) $\{y \in L : \xi_{q,A}(y) < \lambda\} = \lambda q + A - I\!R_{++}q$ *and* $\lambda q + A \subseteq \{y \in L : \xi_{q,A}(y) \leq \lambda\}$; *thus,* $\{y \in L : \xi_{q,A}(y) < 0\} = -I\!R_{++}q + A$ *and* $\{y \in L : \xi_{q,A}(y) < +\infty\} = I\!Rq + A$.

(b) $\lambda q + \text{intA} \subseteq \{y \in L : \xi_{q,A}(y) < \lambda\}$.

(c) $\{y \in L : \xi_{q,A}(y) \leq \lambda\} \subseteq \lambda q + \text{cl}(A - I\!R_{++}q)$.

(d) $\{y \in L : \xi_{q,A}(y) = \lambda\} \subseteq \lambda q + \text{cl}(A - I\!R_{++}q) \setminus (A - I\!R_{++}q)$.

First results concerning the linear approach (Zălinescu (1986), Torre-Popovici-Rocca (2011), Qiu-He (2013), Qiu (2014),. . .)

Zălinescu (1986)

Proposition 1. Let $K \subset X$ be a convex cone and $\bar{x} \in X$. Then $X = K + R\bar{x}$ if and only if

a) K is a linear subspace of codimension 1 and $\bar{x} \notin K$, or

b) $\{\bar{x}, -\bar{x}\} \cap K^i \neq \emptyset$.

Algebraic interior

$$K^i := \{x \in X : \forall\, v \in X, \exists\, \lambda > 0 \text{ s.t. } x + [0, \lambda]v \subset K\}$$

Qiu (2014)

Lemma 3.2. *(Refer to [10,21,40,41,44].) Let $y \in Y$ and $r \in R$. Then we have:*

(i) $\xi_{k_0}(y) < r \Leftrightarrow y \in rk_0 - \text{vint}_{k_o}(D)$.
(ii) $\xi_{k_0}(y) \leq r \Leftrightarrow y \in rk_0 - \text{vcl}_{k_0}(D)$.
(iii) $\xi_{k_0}(y) = r \Leftrightarrow y \in rk_0 - (\text{vcl}_{k_0}(D) \setminus \text{vint}_{k_0}(D))$. *In particular,* $\xi_{k_0}(k_0) = 1$ *and* $\xi_{k_0}(0) = 0$.
(iv) $\xi_{k_0}(y) \geq r \Leftrightarrow y \notin rk_0 - \text{vint}_{k_0}(D)$.
(v) $\xi_{k_0}(y) > r \Leftrightarrow y \notin rk_0 - \text{vcl}_{k_0}(D)$.

Moreover, we have:

(vi) $\xi_{k_0}(y_1 + y_2) \leq \xi_{k_0}(y_1) + \xi_{k_0}(y_2)$, $\forall y_1, y_2 \in Y$.
(vii) $\xi_{k_0}(y + \lambda k_0) = \xi_{k_0}(y) + \lambda$, $\forall y \in Y$, $\forall \lambda \in R$.
(viii) $y_1 \leq_D y_2 \Rightarrow \xi_{k_0}(y_1) \leq \xi_{k_0}(y_2)$.

$$\text{vint}_{k_0} D := D + (0, +\infty)k_0,$$
$$\text{vcl}_{k_0} D := \{x \in X : \forall \lambda > 0\, \exists \lambda' \in [0, \lambda] \text{ s.t. } x + \lambda' k_0 \in D\}$$
$$= D \cup \{x \in X : \exists \lambda > 0 \text{ s.t. } x + (0, \lambda]k_0 \subset D\}$$

Aims of the seminar

1.- To study the smallest strictly monotonic functional in the linear setting

2.- As application, to state an Ekeland variational principle for a vector mapping

Outline

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Preliminaries

Y real linear space

$\emptyset \neq E \subset Y$, $q \in Y \setminus \{0\}$, $\varphi_E^q : Y \to \mathbb{R} \cup \{\pm\infty\}$,

$$\varphi_E^q(y) = \inf\{t \in \mathbb{R} : y \in tq - E\}, \quad \forall y \in Y$$

Parameters: E, q

Algebraic tools

Y real linear space

$\emptyset \neq F \subset Y$, $q \in Y$

Algebraic interior

$$\operatorname{core} F := \{y \in Y : \forall\, v \in Y,\ \exists\, \lambda > 0 \text{ s.t. } y + [0,\lambda]v \subset F\}$$

Vector closure by q

$$\operatorname{vcl}_q F := \{y \in Y : \forall\, \lambda > 0\ \exists\, \lambda' \in [0,\lambda] \text{ s.t. } y + \lambda' q \in F\}$$

Points from which the ray with direction q is not asymptotically contained in $Y \backslash F$

$$\operatorname{ovcl}_q^{+\infty} F := \{y \in Y : \forall\, \lambda > 0\ \exists\, \lambda' \in [\lambda,+\infty) \text{ s.t. } y + \lambda' q \in F\}$$

Properties

$\varphi : Y \to \mathbb{R} \cup \{\pm\infty\}$

$\operatorname{dom} \varphi := \{y \in Y : \varphi(y) < +\infty\}$

$S(\varphi, r, =) := \{y \in Y : \varphi(y) = r\}$, for all $r \in \mathbb{R} \cup \{\pm\infty\}$

φ is proper if $S(\varphi, -\infty, =) = \emptyset$ and $\operatorname{dom} \varphi \neq \emptyset$

Theorem

(i) $\varphi_E^q = \varphi_{\mathrm{vcl}_q E}^q$

(ii) $\operatorname{dom} \varphi_E^q = \mathbb{R}q - E$ and $S(\varphi_E^q, -\infty, =) = \operatorname{ovcl}_q^{+\infty}(-E)$

(iii) φ_E^q is proper if and only if $\operatorname{ovcl}_q^{+\infty}(-E) = \emptyset$

(iv) φ_E^q is finite if and only if $\operatorname{ovcl}_q^{+\infty}(-E) = \emptyset$ and $Y = \mathbb{R}q - E$

Properties

$\varphi : Y \to \mathbb{R} \cup \{\pm\infty\}$

$S(\varphi, r, \mathcal{R}) := \{y \in Y : \varphi(y)\mathcal{R}r\}$, for all $r \in \mathbb{R}$ and for all $\mathcal{R} \in \{\leq, <, \geq, >\}$

Theorem

(v) $\varphi^q_E(y + rq) = \varphi^q_E(y) + r$, for all $y \in Y$ and for all $r \in \mathbb{R}$ (translative function)

(vi) $S(\varphi^q_E, r, \mathcal{R}) = S(\varphi^q_E, 0, \mathcal{R}) + rq$, for all $\mathcal{R} \in \{\leq, <, =, \geq, >\}$ and for all $r \in \mathbb{R}$

(vii) $S(\varphi^q_E, 0, \leq) = (-\infty, 0]q - \mathrm{vcl}_q E$

(viii) $S(\varphi^q_E, 0, <) = (-\infty, 0)q - \mathrm{vcl}_q E$

(ix) $S(\varphi^q_E, 0, =) = (-\mathrm{vcl}_q E) \setminus ((-\infty, 0)q - \mathrm{vcl}_q E)$

(x) $S(\varphi^q_E, 0, \geq) = Y \setminus ((-\infty, 0)q - \mathrm{vcl}_q E)$

Properties

$\emptyset \neq C \subset Y$

$$y_1, y_2 \in Y, \quad y_1 \leq_C y_2 \iff y_1 - y_2 \in -C$$

φ is C-nondecreasing if

$$y_1, y_2 \in Y, y_1 \leq_C y_2, y_1 \neq y_2 \Rightarrow \varphi(y_1) \leq \varphi(y_2)$$

Theorem The following statements are equivalent:

(i) φ_E^q is C-nondecreasing

(ii) $\operatorname{vcl}_q E + C \subset [0, +\infty)q + \operatorname{vcl}_q E$

(iii) $E + C \subset [0, +\infty)q + \operatorname{vcl}_q E$

Properties

φ is C-increasing if

$$y_1, y_2 \in Y, y_1 \leq_C y_2, y_1 \neq y_2 \Rightarrow \varphi(y_1) < \varphi(y_2)$$

Theorem Suppose that φ_E^q is finite. Then the following statements are equivalent:

(i) φ_E^q is C-increasing

(ii) $\text{vcl}_q E + C \setminus \{0\} \subset (0, +\infty) q + \text{vcl}_q E$

Observe that if φ_E^q is C-nondecreasing, then

$$S(\varphi_E^q, -\infty, =) - C \setminus \{0\} \subset S(\varphi_E^q, -\infty, =)$$

$$S(\varphi_E^q, +\infty, =) + C \setminus \{0\} \subset S(\varphi_E^q, +\infty, =)$$

Thus, if φ_E^q is not finite, then it cannot be C-increasing

Smallest strictly monotone functional

If $D \subset Y$ is a solid convex cone and $q \in \text{int}\, D$, then φ_D^q is int $D \cup \{0\}$-increasing, since

$$\text{vcl}_q D + \text{int}\, D = \text{int}\, D,$$
$$(0, +\infty)q + \text{vcl}_q D = \text{int}\, D.$$

In addition,

$$S(\varphi_D^q, 0, <) = (-\infty, 0)q - \text{vcl}_q D = -\text{int}\, D$$

Thefore, for each int $D \cup \{0\}$-increasing function $g : Y \to \mathbb{R}$ at $\bar{y}$ (i.e., $g(y) < g(\bar{y})$ for all $y \leq_{\text{int}\, D \cup \{0\}} \bar{y}$, $y \neq \bar{y}$), we have

$$\{y \in Y : \varphi_D^q(y - \bar{y}) < 0\} = \bar{y} - \text{int}\, D \subset \{y \in Y : g(y) < g(\bar{y})\}.$$

Properties

φ is positively homogeneous if $\varphi(\lambda y) = \lambda\varphi(y)$, for all $y \in Y$ and $\lambda > 0$

φ is convex if $\varphi(\alpha y_1 + (1-\alpha)y_2) \leq \alpha\varphi(y_1) + (1-\alpha)\varphi(y_2)$, for all $y_1, y_2 \in Y$ and $\alpha \in (0,1)$

In the previous definition it is assumed $+\infty - \infty = -\infty + \infty = +\infty$

Theorem $\emptyset \neq E \subset Y$, $q \in Y\backslash\{0\}$

(i) If $\operatorname{vcl}_q E$ is a cone, then φ_E^q is positively homogeneous

(ii) If $\operatorname{vcl}_q E$ is convex, then φ_E^q is convex

If additionally we have that

$$\operatorname{vcl}_q E + (0, +\infty)q \subset \operatorname{vcl}_q E$$

then the following characterizations are true:

(iii) φ_E^q is positively homogeneous if and only if $\operatorname{vcl}_q E$ is a cone

(iv) φ_E^q is convex if and only if $\operatorname{vcl}_q E$ is convex

Ekeland variational principle for a vector mapping

$\emptyset \neq D \subset Y \backslash \{0\}$ convex cone.

$\leq_D$ is a partial order

Theorem $f : X \to Y$, (X, d) is a complete metric space, D is algebraic closed ($\mathrm{vcl}_y D = D$, for all $y \in Y$), $\varepsilon > 0$, $\bar{x} \in X$, $q \in D \backslash (-D)$

Suppose that

$\{x \in X : f(x) \leq_D y\}$ is closed, for all $y \in Y$ (lower semicontinuity condition)

$f(x) \not\leq_D f(\bar{x}) - \varepsilon q$, for all $x \in X$ ($\bar{x}$ approximate solution)

Then, there exists $\hat{x} \in X$ such that

(i) $f(\hat{x}) + \varepsilon d(\bar{x}, \hat{x}) q \leq_D f(\bar{x})$

(ii) $x \in X, f(x) + \varepsilon d(x, \hat{x}) q \leq_D f(\hat{x}) \Rightarrow x = \hat{x}$

Ekeland variational principle for a vector mapping

Theorem $f : X \to Y$, (X,d) is a complete metric space, D is algebraic closed, $\varepsilon > 0$, $\bar{x} \in X$, $q \in D \backslash (-D)$. Suppose that

$\{x \in X : f(x) \leq_D y\}$ is closed, for all $y \in Y$ (lower semicontinuity condition)

$f(x) \not\leq_D f(\bar{x}) - \varepsilon q$, for all $x \in X$ ($\bar{x}$ approximate solution)

Then, there exists $\hat{x} \in X$ such that

(i) $f(\hat{x}) + \varepsilon d(\bar{x},\hat{x})q \leq_D f(\bar{x})$

(ii) $x \in X, f(x) + \varepsilon d(x,\hat{x})q \leq_D f(\hat{x}) \Rightarrow x = \hat{x}$

Proof $\varphi_D^q > -\infty$. Indeed, if $\varphi_D^q(y) = -\infty$ for a point $y \in Y$ then

$$y \in S(\varphi_D^q, -\infty, =) = \operatorname{ovcl}_q^{+\infty}(-D)$$

and there exists a sequence (λ_n) such that $\lambda_n \uparrow +\infty$ and $y + \lambda_n q \in -D$, for all n. Therefore, $q + (1/\lambda_n)y \in -D$, and so $q \in \operatorname{vcl}_y(-D) = -D$, a contradiction

On the other hand, φ_D^q is D-nondecreasing (inclusion $E + C \subset [0,+\infty)q + \operatorname{vcl}_q E$ is fulfilled when $E = C = D$)

Ekeland variational principle for a vector mapping

Proof Moreover,

$$S(\varphi_D^q, 0, \leq) = (-\infty, 0]q - \operatorname{vcl}_q D = -D$$
$$S(\varphi_D^q, 0, <) = (-\infty, 0)q - \operatorname{vcl}_q D = (-\infty, 0)q - D$$

In particular, $\varphi_D^q(0) = 0$. Consider $g : X \to \mathbb{R} \cup \{+\infty\}$,

$$g(x) := \varphi_D^q(f(x) - f(\bar{x})), \quad \forall x \in X$$

Let $x \in X$ and $t \in \mathbb{R}$. We have that

$$g(x) \leq t \iff \varphi_D^q(f(x) - f(\bar{x})) \leq t \iff \varphi_D^q(f(x) - f(\bar{x}) - tq) \leq 0$$
$$\iff f(x) - f(\bar{x}) - tq \leq_D 0 \iff f(x) \leq_D f(\bar{x}) + tq$$

From here, we see that g is proper, lower semicontinuous and $g(\bar{x}) - \varepsilon < g(x)$, for all $x \in X$, since

$$f(x) \not\leq_D f(\bar{x}) - \varepsilon q, \forall x \in X \Rightarrow g(x) > -\varepsilon = g(\bar{x}) - \varepsilon$$

Ekeland variational principle for a vector mapping

Proof By applying the Ekeland variational principle to g and $\bar{x}$ we deduce that there exists a point $\hat{x} \in X$ such that

(i) $g(\hat{x}) + \varepsilon d(\bar{x}, \hat{x}) \leq g(\bar{x})$

(ii) $x \in X, g(x) + \varepsilon d(x, \hat{x}) \leq g(\hat{x}) \Rightarrow x = \hat{x}$

Recall that φ_D^q is D-nondecreasing and

$$g(x) \leq t \iff f(x) \leq_D f(\bar{x}) + tq$$

Then, by (i) we obtain $f(\hat{x}) \leq_D f(\bar{x}) - \varepsilon d(\bar{x}, \hat{x})q$

Analogously, if there exists $x \in X$ such that $f(x) + \varepsilon d(x, \hat{x})q \leq_D f(\hat{x})$, then

$$\begin{aligned} g(x) + \varepsilon d(x, \hat{x}) &= \varphi_D^q(f(x) - f(\bar{x})) + \varepsilon d(x, \hat{x}) \\ &= \varphi_D^q(f(x) - f(\bar{x}) + \varepsilon d(x, \hat{x})q) \\ &\leq \varphi_D^q(f(\hat{x}) - f(\bar{x})) \\ &= g(\hat{x}) \end{aligned}$$

and by (ii) we deduce that $x = \hat{x}$

Nonconvex separation functionals

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Thank you for your attention!

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