Farkas' lemma: some extensions and applications

Miguel Á. Goberna Universidad de Alicante

CMM, Universidad de Chile

Optimization Seminar, 3 de noviembre de 2021

Outline

- Gyula Farkas and Hungarian mathematics

Outline

- Gyula Farkas and Hungarian mathematics
- The classical Farkas' lemma

Outline

- Gyula Farkas and Hungarian mathematics
- The classical Farkas' lemma
- Convex infinite Farkas-type lemmas

Outline

- Gyula Farkas and Hungarian mathematics
- The classical Farkas' lemma
- Convex infinite Farkas-type lemmas
- Relaxed convex infinite Farkas-type lemmas

Outline

- Gyula Farkas and Hungarian mathematics
- The classical Farkas' lemma
- Convex infinite Farkas-type lemmas
- Relaxed convex infinite Farkas-type lemmas
- Vector infinite Farkas-type lemmas

Gyula Farkas and Hungarian mathematics

Gyula Farkas and Hungarian mathematics

- According to MathSciNet, the name "Farkas" appears in the title of 6 of my papers (also in the reviews of 16), all of them related to his lemma.

Gyula Farkas and Hungarian mathematics

- According to MathSciNet, the name “Farkas” appears in the title of 6 of my papers (also in the reviews of 16), all of them related to his lemma.
- The term “Farkas lemma” appears in the title of 93 papers and in the review of 344 (the oldest one dated on 1953), but this is a clear underestimation.

Gyula Farkas and Hungarian mathematics

- According to MathSciNet, the name "Farkas" appears in the title of 6 of my papers (also in the reviews of 16), all of them related to his lemma.
- The term "Farkas lemma" appears in the title of 93 papers and in the review of 344 (the oldest one dated on 1953), but this is a clear underestimation.
- Why so many? Because it was the inception of a new method.

Gyula Farkas and Hungarian mathematics

- According to MathSciNet, the name "Farkas" appears in the title of 6 of my papers (also in the reviews of 16), all of them related to his lemma.
- The term "Farkas lemma" appears in the title of 93 papers and in the review of 344 (the oldest one dated on 1953), but this is a clear underestimation.
- Why so many? Because it was the inception of a new method.
- "Professor H. F. Bohnenblust once told me something about research. He had supervised many successful Ph.D. thesis projects - and a few unsuccessful ones. He said this: 'The unsuccessful projects start with some famous old problem (prove the Riemann hypothesis) and then look for a method to solve it. The successful projects start with some new method and then look for a problem' (...). I hope to convince you that every mathematician should know the Farkas theorem [lemma for us] and should know how to use it" (Joel Franklin, 1983).

- Farkas was firstly trained in a Benedictine monastery near the rural property in which his father was a butler and secondly at a *Gymnasium* in Gyor.

- Farkas was firstly trained in a Benedictine monastery near the rural property in which his father was a butler and secondly at a *Gymnasium* in Gyor.
- He studied physics and chemistry at the University of Pest (unique in Hungary).

- Between 1874 and 1880 he was tutor to the children of the Count of Polgárdi.

Gyula Farkas and Hungarian mathematics

- Between 1874 and 1880 he was tutor to the children of the Count of Polgárdi.
- His publications allowed him to become a professor at the University of Pest and, since 1887 on, at the new University of Kolozsáar (present-day Cluj) -created in 1872 by the Hungarian minister of Religion and Education Eötvös-, better known as 'the Göttingen of the monarchy' (Farkas coincided there with Haar, Riesz, Radó, etc.).

Gyula Farkas and Hungarian mathematics

- Between 1874 and 1880 he was tutor to the children of the Count of Polgárdi.
- His publications allowed him to become a professor at the University of Pest and, since 1887 on, at the new University of Kolozsáar (present-day Cluj) -created in 1872 by the Hungarian minister of Religion and Education Eötvös-, better known as 'the Göttingen of the monarchy' (Farkas coincided there with Haar, Riesz, Radó, etc.).
- The first contemporary mathematics competition for secondary school students was the Eötvös competition held in 1894 (at least, Haar and Radó won the 1st price, as Riesz's brother did).

Gyula Farkas and Hungarian mathematics

- Between 1874 and 1880 he was tutor to the children of the Count of Polgárdi.
- His publications allowed him to become a professor at the University of Pest and, since 1887 on, at the new University of Kolozsáar (present-day Cluj) -created in 1872 by the Hungarian minister of Religion and Education Eötvös-, better known as 'the Göttingen of the monarchy' (Farkas coincided there with Haar, Riesz, Radó, etc.).
- The first contemporary mathematics competition for secondary school students was the Eötvös competition held in 1894 (at least, Haar and Radó won the 1st price, as Riesz's brother did).
- "The winners of these competitions, so to say, overlap with the set of mathematicians and physicists who later became well-respected world figures" (John von Neumann, 1929, who couldn't compete due to WWI).

The classical Farkas' lemma

The classical Farkas' lemma

- We start with scalar optimization problem of the form

$$(P) \quad \min f(x) \text{ s.t. } x \in A,$$

where $A \subset X$ is the *feasible set*, X is the *decision space*, and $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is the *objective function*.

The classical Farkas' lemma

- We start with scalar optimization problem of the form

$$(P) \quad \min f(x) \text{ s.t. } x \in A,$$

where $A \subset X$ is the *feasible set*, X is the *decision space*, and $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is the *objective function*.

- $\overline{x} \in A$ is a *minimizer* of (P), $\overline{x} \in \mathrm{sol}(P)$ in short, when

$$x \in A \Rightarrow f(\overline{x}) \leq f(x)$$

or, equivalently, denoting $h(x) := f(\overline{x}) - f(x)$,

$$A \subset B := [h \leq 0] := \{x \in X : h(x) \leq 0\}$$

The classical Farkas' lemma

- We start with scalar optimization problem of the form

$$(P) \quad \min f(x) \text{ s.t. } x \in A,$$

  where $A \subset X$ is the *feasible set*, X is the *decision space*, and $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is the *objective function*.

- $\overline{x} \in A$ is a *minimizer* of (P), $\overline{x} \in \mathrm{sol}(P)$ in short, when

$$x \in A \Rightarrow f(\overline{x}) \leq f(x)$$

  or, equivalently, denoting $h(x) := f(\overline{x}) - f(x)$,

$$A \subset B := [h \leq 0] := \{x \in X : h(x) \leq 0\}$$

- A Farkas-type lemma is a characterization of the *set containment* $A \subset B$, when B is a sublevel set of some function.

The classical Farkas' lemma

Let $A = \{x \in \mathbb{R}^n : \langle a_t, x \rangle \leq b_t, t = 1, ..., m\}$.

Lemma (Non-homogeneous Farkas)

If $A \neq \emptyset$ and $(a, b) \in \mathbb{R}^{n+1}$, then,

$$A \subset \{x \in \mathbb{R}^n : \langle a, x \rangle \leq b\}$$
$$\Leftrightarrow \begin{pmatrix} a \\ b \end{pmatrix} \in \operatorname{cone}\left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t = 1, ..., m; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\}$$

The cone above is the *characteristic cone* of this system. From this affine/affine Farkas lemma it is easy to prove the four pillars of LP theory.

Theorem (Existence)

$$A \neq \emptyset \Leftrightarrow \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \notin \operatorname{cone}\left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t = 1, ..., m; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\}$$

The classical Farkas' lemma

We now consider the LP problem

$$(LP) \quad \min \ \langle c, x\rangle \ \text{s.t.} \ x \in A =: \{\langle a_t, x\rangle \leq b_t, t = 1, ..., m\}.$$

The *set of active indices* and the *active cone* at $\overline{x} \in A$ are

$$T(\overline{x}) = \{t \in T := \{1, ..., m\} : \langle a_t, \overline{x}\rangle = b_t\}$$

$$\operatorname{cone}\{a_t, t \in T(\overline{x})\}$$

Theorem (LP Optimality Theorem)

Let $\overline{x} \in A$. Then $\overline{x} \in \operatorname{sol}(LP)$ iff

$$-c \in \operatorname{cone}\{a_t, t \in T(\overline{x})\}, \tag{1}$$

(geometric form of the KKT condition).

The classical Farkas' lemma

The *dual problem* of (LP) consists in the maximization of lower bounds for $\{\langle c, x\rangle : x \in A\}$:

$$(LD): \quad \sup_{\lambda \in \mathbb{R}_+^m} \quad -\sum_{t=1}^{m} \lambda_t b_t$$

$$\text{s.t.} \quad \sum_{t=1}^{m} \lambda_t a_t = -c.$$

Theorem (LP Duality Theorem)

If (LP) is bounded, i.e., $\inf (LP) \in \mathbb{R}$, then (LP) and (LD) are both solvable and their optimal values coincide, i.e., $\min (LP) = \max (LD)$.

The classical Farkas' lemma

Theorem (Motzkin)

Any polyhedral convex set is the sum of a polytope with a polyhedral convex cone.

Role of Farkas' lemma in the proofs

(see, e.g., G-Jornet-Puente, 2004)

$$\text{Existence theor.} \Longleftarrow \text{Farkas lemma} \Longrightarrow \text{Optimality theor.}$$

$$\Downarrow$$

$$\text{Motzkin theorem}$$

$$\Downarrow$$

$$\text{Duality theorem}$$

The classical Farkas' lemma

The next famous result was known by Karush, 1936, but it was first published by Kuhn-Tucker, 1951. Let

$$(NLP) \quad \min f(x) \text{ s.t. } g_t(x) \leq 0, \ t = 1, ..., m,$$

where all functions are $\mathcal{C}^1$. Let A be the feasible set of (P) and

$$T(\overline{x}) := \{ t \in \{1, ..., m\} : g_t(\overline{x}) = 0\}$$

be the set of active indices at $\overline{x} \in A$.

Theorem (NLP Optimality Theorem)

If $\overline{x}$ is a local minimum of (NLP) and $\{\nabla g_t(\overline{x}), t \in T(\overline{x})\}$ is linearly independent, then $\exists \lambda_t \geq 0$, $t \in T(\overline{x})$, such that

$$-\nabla f(\overline{x}) \in \operatorname{cone}\{\nabla g_t(\overline{x}), t \in T(\overline{x})\}$$

(geometric form of the KKT condition).

The classical Farkas' lemma

- **Sketch of proof** (see, e.g., G-Jornet-Puente, 2004, and Aragón-G-López-Rodriguez, 2019):

The classical Farkas' lemma

- **Sketch of proof** (see, e.g., G-Jornet-Puente, 2004, and Aragón-G-López-Rodriguez, 2019):
- Under the linear independence assumption, it can be shown that

$$\{x \in \mathbb{R}^n : \langle \nabla g_t(\overline{x}), x \rangle \leq 0, t \in T(\overline{x})\}$$
$$\subset \{x \in \mathbb{R}^n : \langle \nabla f(\overline{x}), x \rangle \geq 0\}$$

The classical Farkas' lemma

- **Sketch of proof** (see, e.g., G-Jornet-Puente, 2004, and Aragón-G-López-Rodriguez, 2019):
- Under the linear independence assumption, it can be shown that
$$\{x \in \mathbb{R}^n : \langle \nabla g_t(\overline{x}), x \rangle \leq 0, t \in T(\overline{x})\} \subset \{x \in \mathbb{R}^n : \langle \nabla f(\overline{x}), x \rangle \geq 0\}$$
- Then, applying the homogeneous Farkas lemma to the above inclusion $A \subset B$, one gets

$$-\nabla f(\overline{x}) \in \text{cone}\{\nabla g_t(\overline{x}), t \in T(\overline{x})\}$$

□

A bit of history

- 1838: Ostrogradski tries to prove the NLP optimality theorem (with f being the potential function of a conservative field), but omits the constraint qualification and the characterization of the inclusion

$$A := \{x \in \mathbb{R}^n : \langle a_t, x \rangle \leq 0, t = 1, ..., m\} \subset [\langle a, \cdot \rangle \leq 0]$$

A bit of history

- 1838: Ostrogradski tries to prove the NLP optimality theorem (with f being the potential function of a conservative field), but omits the constraint qualification and the characterization of the inclusion

$$A := \{x \in \mathbb{R}^n : \langle a_t, x\rangle \leq 0, t = 1, ..., m\} \subset [\langle a, \cdot\rangle \leq 0]$$

- 1894: The physicist Farkas observes the 2nd omission (his future lemma).

The classical Farkas' lemma

A bit of history

- 1838: Ostrogradski tries to prove the NLP optimality theorem (with f being the potential function of a conservative field), but omits the constraint qualification and the characterization of the inclusion

$$A := \{x \in \mathbb{R}^n : \langle a_t, x \rangle \leq 0, t = 1, ..., m\} \subset [\langle a, \cdot \rangle \leq 0]$$

- 1894: The physicist Farkas observes the 2nd omission (his future lemma).
- 1902: Farkas gives the 1st correct proof of the linear/linear (or homogeneous) Farkas' lemma:

$$A \subset \{x \in \mathbb{R}^n : \langle a, x \rangle \leq 0\}$$
$$\Leftrightarrow a \in \operatorname{cone}\{a_t, t = 1, ..., m\}$$

A bit of history

- 1838: Ostrogradski tries to prove the NLP optimality theorem (with f being the potential function of a conservative field), but omits the constraint qualification and the characterization of the inclusion

$$A := \{x \in \mathbb{R}^n : \langle a_t, x \rangle \leq 0, t = 1, ..., m\} \subset [\langle a, \cdot \rangle \leq 0]$$

- 1894: The physicist Farkas observes the 2nd omission (his future lemma).
- 1902: Farkas gives the 1st correct proof of the linear/linear (or homogeneous) Farkas' lemma:

$$A \subset \{x \in \mathbb{R}^n : \langle a, x \rangle \leq 0\}$$
$$\Leftrightarrow a \in \operatorname{cone} \{a_t, t = 1, ..., m\}$$

- 1911: Minkowski proves the affine/affine (also called non-homogeneous) Farkas' lemma.

Some applications of the classical Farkas' lemma

1. Existence, optimality and duality in LP (as we have already seen).

Some applications of the classical Farkas' lemma

1. Existence, optimality and duality in LP (as we have already seen).
2. Optimality in NLP (as we have already seen, too).

Some applications of the classical Farkas' lemma

1. Existence, optimality and duality in LP (as we have already seen).
2. Optimality in NLP (as we have already seen, too).
3. Machine learning (Mangasarian, 2000 and 2002): the incorporation of prior knowledge in the form of a polyhedral knowledge set A in the construction of a linear classifier is modelled as the set containment $A \subset [\langle a, \cdot \rangle \leq b]$.

Some applications of the classical Farkas' lemma

1. Existence, optimality and duality in LP (as we have already seen).
2. Optimality in NLP (as we have already seen, too).
3. Machine learning (Mangasarian, 2000 and 2002): the incorporation of prior knowledge in the form of a polyhedral knowledge set A in the construction of a linear classifier is modelled as the set containment $A \subset [\langle a, \cdot \rangle \leq b]$.
4. Probability (Franklin, 1983): moment problems and Markov chains.

Some applications of the classical Farkas' lemma

1. Existence, optimality and duality in LP (as we have already seen).
2. Optimality in NLP (as we have already seen, too).
3. Machine learning (Mangasarian, 2000 and 2002): the incorporation of prior knowledge in the form of a polyhedral knowledge set A in the construction of a linear classifier is modelled as the set containment $A \subset [\langle a, \cdot \rangle \leq b]$.
4. Probability (Franklin, 1983): moment problems and Markov chains.
5. Economics (Franklin, 1983).

Some applications of the classical Farkas' lemma

1. Existence, optimality and duality in LP (as we have already seen).
2. Optimality in NLP (as we have already seen, too).
3. Machine learning (Mangasarian, 2000 and 2002): the incorporation of prior knowledge in the form of a polyhedral knowledge set A in the construction of a linear classifier is modelled as the set containment $A \subset [\langle a, \cdot \rangle \leq b]$.
4. Probability (Franklin, 1983): moment problems and Markov chains.
5. Economics (Franklin, 1983).
6. Finance (Elliott-Kopp, 2005; Prékopa, 2006).

Some applications of the classical Farkas' lemma

1. Existence, optimality and duality in LP (as we have already seen).
2. Optimality in NLP (as we have already seen, too).
3. Machine learning (Mangasarian, 2000 and 2002): the incorporation of prior knowledge in the form of a polyhedral knowledge set A in the construction of a linear classifier is modelled as the set containment $A \subset [\langle a, \cdot \rangle \leq b]$.
4. Probability (Franklin, 1983): moment problems and Markov chains.
5. Economics (Franklin, 1983).
6. Finance (Elliott-Kopp, 2005; Prékopa, 2006).
7. Robotics (Robotka-Vizvari, 2006).

Extensions in the finite setting
(i.e., with A being solution set of a system of finitely many variables and constraints)

- $A \subset B := [h \leq 0]$, only with weak inequalities: Jeyakumar (2001) and Dinh-Jeyakumar (2014) present selections of Farkas-type lemmas for conic-linear, conic-sublinear and convex inequality systems, conic-convex systems, classes of non-convex systems such as DC systems, composite convex systems and quadratic systems, and inequality systems involving uncertain vectors, matrices, and polynomials.

Extensions in the finite setting
(i.e., with A being solution set of a system of finitely many variables and constraints)

- $A \subset B := [h \leq 0]$, only with weak inequalities: Jeyakumar (2001) and Dinh-Jeyakumar (2014) present selections of Farkas-type lemmas for conic-linear, conic-sublinear and convex inequality systems, conic-convex systems, classes of non-convex systems such as DC systems, composite convex systems and quadratic systems, and inequality systems involving uncertain vectors, matrices, and polynomials.
- Similar, with strict inequalities (and maybe weak ones): results can be found in the source book Fajardo-G-Rodriguez-Vicente_Pérez (2020) as well as in Rodriguez-Vicente_Pérez (2021), which also contains an existence theorem for convex systems.

The classical Farkas' lemma

- The containment problem consists in characterizing, in terms of the data, the inclusion $A \subset B$, where the *contained* A and the *container* B are solution sets of systems:

The classical Farkas' lemma

- The containment problem consists in characterizing, in terms of the data, the inclusion $A \subset B$, where the *contained* A and the *container* B are solution sets of systems:
- It was solved by Mangasarian (2000, 2002) for linear and convex-reverse convex quadratic systems with finitely many variables and constraints.

Other contributions to the containment problem:

- Jeyakumar (2003), for converse and reverse convex systems.

The classical Farkas' lemma

Other contributions to the containment problem:

- Jeyakumar (2003), for converse and reverse convex systems.
- G-Jeyakumar-Dinh (2006), for linear systems containing strict inequalities.

The classical Farkas' lemma

Other contributions to the containment problem:

- Jeyakumar (2003), for converse and reverse convex systems.
- G-Jeyakumar-Dinh (2006), for linear systems containing strict inequalities.
- Doagooei-Mohebi (2009), for systems of cone-convex inequalities.

The classical Farkas' lemma

Other contributions to the containment problem:

- Jeyakumar (2003), for converse and reverse convex systems.
- G-Jeyakumar-Dinh (2006), for linear systems containing strict inequalities.
- Doagooei-Mohebi (2009), for systems of cone-convex inequalities.
- Suzuki-Kuroiwa (2009, 2011), for quasiconvex systems.

Other contributions to the containment problem:

- Jeyakumar (2003), for converse and reverse convex systems.
- G-Jeyakumar-Dinh (2006), for linear systems containing strict inequalities.
- Doagooei-Mohebi (2009), for systems of cone-convex inequalities.
- Suzuki-Kuroiwa (2009, 2011), for quasiconvex systems.
- Suzuki (2010), for quasiconvex systems containing strict inequalities.

Other contributions to the containment problem:

- Jeyakumar (2003), for converse and reverse convex systems.
- G-Jeyakumar-Dinh (2006), for linear systems containing strict inequalities.
- Doagooei-Mohebi (2009), for systems of cone-convex inequalities.
- Suzuki-Kuroiwa (2009, 2011), for quasiconvex systems.
- Suzuki (2010), for quasiconvex systems containing strict inequalities.
- Kellner-Theobald-Trabandt (2013), for spectrahedra.

Other contributions to the containment problem:

- Jeyakumar (2003), for converse and reverse convex systems.
- G-Jeyakumar-Dinh (2006), for linear systems containing strict inequalities.
- Doagooei-Mohebi (2009), for systems of cone-convex inequalities.
- Suzuki-Kuroiwa (2009, 2011), for quasiconvex systems.
- Suzuki (2010), for quasiconvex systems containing strict inequalities.
- Kellner-Theobald-Trabandt (2013), for spectrahedra.
- Jeyakumar-Lee-Lee (2016), for SOS-convex polynomial systems.

Extensions to the linear semi-infinite setting

- A Farkas-type lemma involving the set

$$A = \{x \in X : g_t(x) \leq 0, t \in T\}$$

is called semi-infinite when card T or dim X is infinite, but not both.

Extensions to the linear semi-infinite setting

- A Farkas-type lemma involving the set

$$A = \{x \in X : g_t(x) \leq 0, t \in T\}$$

  is called semi-infinite when card T or dim X is infinite, but not both.

- $|\text{card } T| < \infty$ and $\dim X = \infty$: In 1924 Haar considered $X = \mathcal{C}([0,1])$, $\langle a, x\rangle = \int_0^1 a(s)\,x(s)\,ds$, $\forall a, x \in X$, and $A = \{x \in X : \langle a_t, x\rangle \leq 0, t \in T\}$ with T finite. He proved the following linear/linear Farkas lemma: If $a \in X$ and $\{a_t, t \in T\} \subset X$ is linearly independent, then

$$A \subset \{x \in X : \langle a, x\rangle \leq 0\}$$
$$\Leftrightarrow a \in \text{cone}\{a_t, t \in T\}$$

The classical Farkas' lemma

- $|\mathrm{card}\, T| = \infty$ and $\dim X = n$: G-López-Pastor (1981) "rediscovered" the following basic result for LSIP theory (see G-López, 1998): if $(a, b) \in \mathbb{R}^{n+1}$ and $A := \{x \in \mathbb{R}^n : \langle a_t, x\rangle \leq b_t, t \in T\} \neq \emptyset$, then,

$$A \subset \{x \in \mathbb{R}^n : \langle a, x\rangle \leq b\}$$
$$\Leftrightarrow \begin{pmatrix} a \\ b \end{pmatrix} \in \mathrm{cl\,cone}\left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\}$$

- $|\mathrm{card}\, T| = \infty$ and $\dim X = n$: G-López-Pastor (1981) "rediscovered" the following basic result for LSIP theory (see G-López, 1998): if $(a, b) \in \mathbb{R}^{n+1}$ and $A := \{x \in \mathbb{R}^n : \langle a_t, x\rangle \leq b_t, t \in T\} \neq \emptyset$, then,

$$A \subset \{x \in \mathbb{R}^n : \langle a, x\rangle \leq b\}$$
$$\Leftrightarrow \begin{pmatrix} a \\ b \end{pmatrix} \in \mathrm{cl\,cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T; \begin{pmatrix} 0_n \\ 1 \end{pmatrix}\right\}$$

- This affine/affine Farkas' lemma was actually the LSIP version of Theorem 2 in Chu (1966), on LIP:

Duality theorems in LSIP

(mostly proved in G-López (1998) thanks to Farkas' lemma.)

A linear system satisfies the *FMCQ* when its characteristic cone is closed and a set is called Motzkin decomposable if it is the sum of a compact convex set with a closed convex cone (these sets have been characterized by G, Gutiérrez, Iusem, Martínez Legaz and Todorov in 2010-2013). Let $\inf(LSIP), \sup(LSID) \in \mathbb{R}$.

- Strong duality: If FMCQ holds, then $\inf(LSIP) = \max(LSID)$.

Duality theorems in LSIP

(mostly proved in G-López (1998) thanks to Farkas' lemma.)

A linear system satisfies the *FMCQ* when its characteristic cone is closed and a set is called Motzkin decomposable if it is the sum of a compact convex set with a closed convex cone (these sets have been characterized by G, Gutiérrez, Iusem, Martínez Legaz and Todorov in 2010-2013). Let $\inf(LSIP), \sup(LSID) \in \mathbb{R}$.

- Strong duality: If FMCQ holds, then $\inf(LSIP) = \max(LSID)$.
- Reverse strong duality: If $-c \in \mathrm{ri}\,\mathrm{cone}\{a_t, t \in T\}$, then $\min(LSIP) = \sup(LSID)$ and $\mathrm{sol}(LSIP)$ is the sum of a non-empty compact convex set with a linear subspace (from G-López-Volle, 2014, dealing with CIP).

Duality theorems in LSIP

(mostly proved in G-López (1998) thanks to Farkas' lemma.)
A linear system satisfies the *FMCQ* when its characteristic cone is closed and a set is called Motzkin decomposable if it is the sum of a compact convex set with a closed convex cone (these sets have been characterized by G, Gutiérrez, Iusem, Martínez Legaz and Todorov in 2010-2013). Let $\inf(LSIP), \sup(LSID) \in \mathbb{R}$.

- Strong duality: If FMCQ holds, then $\inf(LSIP) = \max(LSID)$.
- Reverse strong duality: If $-c \in \operatorname{ri} \operatorname{cone}\{a_t, t \in T\}$, then $\min(LSIP) = \sup(LSID)$ and $\operatorname{sol}(LSIP)$ is the sum of a non-empty compact convex set with a linear subspace (from G-López-Volle, 2014, dealing with CIP).
- Perfect duality: If FMCQ holds and either $-c \in \operatorname{ri} \operatorname{cone}\{a_t, t \in T\}$ or A is Motzkin decomposable, then $\min(LSIP) = \max(LSID)$.

Application to the polynomial representation of polyhedra
We show the idea of Theorem 4.1 in G-Jornet-Puente-Todorov (1999) with the square $[0,1]^2$. The characteristic cone of $\{0 \leq x_i \leq 1, i = 1, 2\}$ is

$$\mathrm{cone}\left\{\left(-1,0,0\right),\left(1,0,1\right),\left(0,-1,0\right),\left(0,1,1\right)\right\} = \mathrm{cone}\,\gamma,$$

where γ is the Lagrange-like interpolating curve through the 4 generators:

Convex infinite Farkas-type lemmas

Convex infinite Farkas-type lemmas

Denote by $\Gamma(X)$ the set of proper lsc convex functions on X (a lcHtvs). Recall that the Fenchel-Moreau conjugate of $h \in \Gamma(X)$,

$$h^*(u) := \sup\{\langle u, x\rangle - h(x) : x \in \mathrm{dom}\ h\}$$

satisfies $h^{**} = h$, so that any element of $\Gamma(X)$ is the supremum of continuous affine functions.

Theorem (convex/convex Farkas-type lemma)

Let $f, g_t \in \Gamma(X)$, and $A = \{x \in X : g_t(x) \leq 0, t \in T\} \neq \emptyset$. Then,

$$A \subset [f \leq 0] \Leftrightarrow \mathrm{epi} f^* \subset \mathrm{cl\,cone}\left\{\bigcup_{t \in T} \mathrm{epi} g_t^*\right\}$$

As in LSIP, $\operatorname{cone}\{\bigcup_{t\in T}\operatorname{epi} g_t^*\}$ is called *characteristic cone.* If it is w^*-closed, we say that the *FMCQ* holds.
We proved it in Dinh-G-López (2006) by linearizing the functions g_t to apply Theorem 2 of Chu (1966)

Corollary (Afine/affine Farkas lemma)

Let $a^*, a_t^* \in X^*$, $b_t, b \in \mathbb{R}$, *such that* $A = \{x \in X : \langle a_t^*, x\rangle \leq b_t, t \in T\} \neq \emptyset$. *Then,*

$$A \subset \{x \in X : \langle a^*, x\rangle \leq b\}$$
$$\Leftrightarrow (a^*, b) \in \operatorname{cl}\operatorname{cone}\{(a_t^*, b_t), t \in T;\ (0,1)\}$$

The characteristic cone in LIP is $\operatorname{cone}\{(a_t^*, b_t), t \in T;\ (0,1)\}$.

Application to linear infinite programming
In Dinh-G-López-Volle (2020) we apply this corollary to the LIP problem

$$(LIP) \quad \inf \langle c^*, x\rangle \ \text{ s.t. } \ \langle a_t^*, x\rangle \leq b_t, t \in T,$$

whose Haar's dual problem is

$$(LID) \qquad \sup_{S\in\mathcal{F}(T),\lambda\in\mathbb{R}_+^S} \left\{ -\sum_{t\in\operatorname{supp}\lambda} \lambda_t b_t : -\sum_{t\in\operatorname{supp}\lambda} \lambda_t a_t^* = c^* \right\},$$

where

$$\mathcal{F}(T) := \{S \in \mathcal{P}(T) : 1 \leq |S| < +\infty\}$$

and the support of λ, $\operatorname{supp}\lambda$, is formed by the indices of non-zero coordinates of λ.

Convex infinite Farkas-type lemmas

- $\inf(LIP) = \max(LID) \in \mathbb{R} \Longrightarrow (LIP)$ is *reducible,* i.e., $\exists S \in \mathcal{F}(T)$:

$$\inf(LIP) = \inf\{\langle c^*, x\rangle : \langle a_t^*, x\rangle \leq b_t, t \in S\}$$

- $\inf(LIP) = \max(LID) \in \mathbb{R} \Longrightarrow (LIP)$ is *reducible,* i.e., $\exists S \in \mathcal{F}(T)$:

$$\inf(LIP) = \inf\{\langle c^*, x\rangle : \langle a_t^*, x\rangle \leq b_t, t \in S\}$$

- The converse statement holds if $\{\langle a_t^*, x\rangle \leq b_t, \ \forall t \in S\}$ satisfies FMCQ.

Convex infinite Farkas-type lemmas

- $\inf(LIP) = \max(LID) \in \mathbb{R} \Longrightarrow (LIP)$ is *reducible,* i.e., $\exists S \in \mathcal{F}(T)$:

$$\inf(LIP) = \inf\{\langle c^*, x\rangle : \langle a_t^*, x\rangle \leq b_t, t \in S\}$$

- The converse statement holds if $\{\langle a_t^*, x\rangle \leq b_t, \ \forall t \in S\}$ satisfies FMCQ.
- $\inf(LIP) = \sup(LID) \in \mathbb{R} \Longrightarrow (LIP)$ is *discretizable,* i.e., $\exists (S_r)_{r\in\mathbb{N}} \subset \mathcal{F}(T)$:

$$\inf(LIP) = \lim_r \inf\{\langle c^*, x\rangle : \langle a_t^*, x\rangle \leq b_t, t \in S_r\}$$

Convex infinite Farkas-type lemmas

- $\inf(LIP) = \max(LID) \in \mathbb{R} \Longrightarrow (LIP)$ is *reducible,* i.e., $\exists S \in \mathcal{F}(T)$:

$$\inf(LIP) = \inf\{\langle c^*, x\rangle : \langle a_t^*, x\rangle \leq b_t, t \in S\}$$

- The converse statement holds if $\{\langle a_t^*, x\rangle \leq b_t, \ \forall t \in S\}$ satisfies FMCQ.
- $\inf(LIP) = \sup(LID) \in \mathbb{R} \Longrightarrow (LIP)$ is *discretizable,* i.e., $\exists (S_r)_{r\in\mathbb{N}} \subset \mathcal{F}(T)$:

$$\inf(LIP) = \lim_r \inf\{\langle c^*, x\rangle : \langle a_t^*, x\rangle \leq b_t, t \in S_r\}$$

- Both converse statements hold if $X = \mathbb{R}^n$ (i.e., in LSIP).

Convex infinite Farkas-type lemmas

From the LIP Farkas-type lemma (or from the existence theorem for linear systems), one can obtain the following existence theorem (Dinh-G-López, 2006).

Theorem (Existence theorem for convex systems)

Let $A = \{x \in X : g_t(x) \leq 0, t \in T\}$, *with* $g_t \in \Gamma(X), \forall t \in T$. *Then,*

$$A \neq \emptyset \iff (0_{X^*}, -1) \notin \operatorname{cl}\operatorname{cone}\left\{\bigcup_{t \in T} \operatorname{epi} g_t^*\right\}$$

- Consider now the convex infinite problem

$$(CIP) \quad \min f(x) \ \text{ s.t. } \ x \in A =: \{x \in X : g_t(x) \leq 0, t \in T\}$$

- Consider now the convex infinite problem

$$(CIP) \quad \min f(x) \text{ s.t. } x \in A =: \{x \in X : g_t(x) \leq 0, t \in T\}$$

- Set constraints of the form $x \in C$, where $\emptyset \neq C \subset X$ is closed and convex, can be aggregated to (CIP) through the indicator function $i_C \in \Gamma(X)$.

Convex infinite Farkas-type lemmas

- Consider now the convex infinite problem

$$(CIP) \quad \min f(x) \ \text{ s.t. } \ x \in A =: \{x \in X : g_t(x) \leq 0, t \in T\}$$

- Set constraints of the form $x \in C$, where $\emptyset \neq C \subset X$ is closed and convex, can be aggregated to (CIP) through the indicator function $i_C \in \Gamma(X)$.
- $\overline{x} \in \mathrm{sol}(CIP)$ iff

$$A \subset [f(\overline{x}) - f \leq 0]$$

Convex infinite Farkas-type lemmas

- Consider now the convex infinite problem

$$(CIP) \quad \min f(x) \text{ s.t. } x \in A =: \{x \in X : g_t(x) \leq 0, t \in T\}$$

- Set constraints of the form $x \in C$, where $\emptyset \neq C \subset X$ is closed and convex, can be aggregated to (CIP) through the indicator function $i_C \in \Gamma(X)$.
- $\overline{x} \in \mathrm{sol}(CIP)$ iff
$$A \subset [f(\overline{x}) - f \leq 0]$$
- Since $f(\overline{x}) - f$ is a concave function, the optimality theorem requires a convex/reverse-convex Farkas' lemma.

The next two Farkas-type lemmas were proved in Dinh-G-López-Son (2007).

Theorem (Convex/reverse convex Farkas' lemma)

Let $f, g_t \in \Gamma(X)$, *and* $A = \{x \in X : g_t(x) \leq 0, t \in T\} \neq \emptyset$. *If FMCQ holds, then,*

$$A \subset [f \geq 0] \Leftrightarrow 0_{X^* \times \mathbb{R}} \in \mathrm{cl}\left(\mathrm{epi} f^* + \mathrm{cone}\left\{\bigcup_{t \in T} \mathrm{epi} g_t^*\right\}\right)$$

- The following closedness data qualification (CDQ) was introduced by Burachik-Jeyakumar (2005):

$$\operatorname{epi} f^* + \operatorname{cl}\operatorname{cone}\left\{\bigcup_{t\in T} \operatorname{epi} g_t^*\right\} \text{ is } w^*\text{-closed}$$

Convex infinite Farkas-type lemmas

- The following closedness data qualification (CDQ) was introduced by Burachik-Jeyakumar (2005):

$$\operatorname{epi} f^* + \operatorname{cl} \operatorname{cone} \left\{ \bigcup_{t \in T} \operatorname{epi} g_t^* \right\} \text{ is } w^*\text{-closed}$$

- Each of the following conditions implies CDQ:

Convex infinite Farkas-type lemmas

- The following closedness data qualification (CDQ) was introduced by Burachik-Jeyakumar (2005):

$$\operatorname{epi} f^* + \operatorname{cl}\operatorname{cone}\left\{\bigcup_{t\in T}\operatorname{epi} g_t^*\right\} \text{ is } w^*\text{-closed}$$

- Each of the following conditions implies CDQ:

1. $\operatorname{epi} f^* + \operatorname{cone}\left\{\bigcup_{t\in T}\operatorname{epi} g_t^*\right\}$ is w^*-closed.

Convex infinite Farkas-type lemmas

- The following closedness data qualification (CDQ) was introduced by Burachik-Jeyakumar (2005):

$$\operatorname{epi} f^* + \operatorname{cl}\operatorname{cone}\left\{ \bigcup_{t\in T} \operatorname{epi} g_t^* \right\} \text{ is } w^*\text{-closed}$$

- Each of the following conditions implies CDQ:

1. $\operatorname{epi} f^* + \operatorname{cone}\left\{\bigcup_{t\in T} \operatorname{epi} g_t^*\right\}$ is w^*-closed.
2. FMCQ holds and f is linear.

Convex infinite Farkas-type lemmas

- The following closedness data qualification (CDQ) was introduced by Burachik-Jeyakumar (2005):

$$\operatorname{epi} f^* + \operatorname{cl} \operatorname{cone} \left\{ \bigcup_{t \in T} \operatorname{epi} g_t^* \right\} \text{ is } w^*\text{-closed}$$

- Each of the following conditions implies CDQ:

1. $\operatorname{epi} f^* + \operatorname{cone} \left\{ \bigcup_{t \in T} \operatorname{epi} g_t^* \right\}$ is w^*-closed.
2. FMCQ holds and f is linear.
3. FMCQ holds and f is continuous at some point of A.

Convex infinite Farkas-type lemmas

Corollary (Convex/reverse convex Farkas' lemma)

Let $f, g_t \in \Gamma(X)$, and $A = \{x \in X : g_t(x) \leq 0, t \in T\} \neq \emptyset$. If CDQ holds, then,

$$A \subset [f \geq 0] \Leftrightarrow 0_{X^* \times \mathbb{R}} \in \operatorname{epi} f^* + \operatorname{cone}\left\{\bigcup_{t \in T} \operatorname{epi} g_t^*\right\}$$

- New Farkas-type lemmas with weaker (generally local) DQs and extensions to non-convex comvex/non-reverse convex functions, appeared in Dinh-Vallet-Nghia (2008), Li-Ng-Pong (2008), Jeyakumar-Li (2009), Fang-Li-Ng (2009, 2010), Dinh-G-López-Volle (2010), Dinh-Mordukhovich-Nghia (2010), Fang-Zhao (2016), Wei-Yao (2018), etc.

Convex infinite Farkas-type lemmas

The elimination of the "cl" operator in the latter Farkas lemma allowed us to get optimality conditions for CIP and a saddle point theorem (in Dinh-G-López-Son, 2007). Here, $\mathbb{R}_+^{(T)}$ denotes the positive cone in the space $\mathbb{R}^{(T)}$ of functions $\lambda : T \longrightarrow \mathbb{R}$ with finite support $\operatorname{supp} \lambda$.

Theorem (CIP optimality theorem)

Assume that FMCQ and CDQ hold, and let $\overline{x} \in A \cap \operatorname{dom} f$. Then $\overline{x} \in \operatorname{sol}(CIP)$ iff $\exists \lambda \in \mathbb{R}_+^{(T)}$ such that

$$\partial g_t(\overline{x}) \neq \emptyset, \ \forall t : \lambda_t > 0,$$

$$0_{X^*} \in \partial f(\overline{x}) + \sum_{t \in T} \lambda_t \partial g_t(\overline{x})$$

$$\lambda_t g_t(\overline{x}) = 0, \ \forall t \in T$$

Convex infinite Farkas-type lemmas

The *Haar-Lagrange function* of (CIP) is

$$L(x,\lambda) := \begin{cases} f(x) + \sum_{t\in T} \lambda_t f_t(x), & \text{if } \lambda \in \mathbb{R}_+^{(T)}, \\ +\infty, & \text{otherwise.} \end{cases}$$

So, the dual problem of (CIP) is

$$(CID) \quad \sup \quad \inf_{x\in X} L(x,\lambda), \text{ s.t. } \quad \lambda \in \mathbb{R}_+^{(T)}$$

Theorem (Saddle point)

Assume that FMCQ and CDQ hold. Then a point $\overline{x} \in \mathrm{sol}(CIP)$ iff $\exists \bar{\lambda} \in \mathbb{R}_+^{(T)}$ such that $(\overline{x}, \bar{\lambda})$ is a saddle point of L, i.e.,

$$L(\overline{x},\lambda) \leq L(\overline{x},\bar{\lambda}) \leq L(x,\bar{\lambda}), \quad \forall \lambda \in \mathbb{R}_+^{(T)}$$

In this case, $\bar{\lambda} \in \mathrm{sol}(CID)$.

Convex infinite Farkas-type lemmas

Theorem (Strong duality)

If $\inf(CIP) \in \mathbb{R}$ *and FMCQ and CDQ hold, then*

$$\inf(CIP) = \max(CID)$$

Role of Farkas' lemma in the proofs
(Dinh-G-López, 2006, and Dinh-G-López-Son, 2007)

Affine/affine Farkas lemma	$\Longrightarrow$	Conv/conv Farkas lemma	$\Longrightarrow$	Existence theor.
		...		
Strong duality theor.	$\Longleftarrow$	Conv/rev. conv. Farkas lemma	$\Longrightarrow$	Saddle point theor.
		$\Downarrow$		
		Optimality theor.		

Relaxed convex infinite Farkas-type lemmas

Relaxed convex infinite Farkas-type lemmas

- Consider again the convex infinite problem of the previous section:

$$(CIP) \quad \min f(x) \text{ s.t. } x \in A =: \{x \in X : g_t(x) \leq 0, t \in T\}$$

Relaxed convex infinite Farkas-type lemmas

- Consider again the convex infinite problem of the previous section:

$$(CIP) \quad \min f(x) \text{ s.t. } x \in A =: \{x \in X : g_t(x) \leq 0, t \in T\}$$

- In Dinh-G-López-Volle (2021a,b) we associate with a given family $\emptyset \neq \mathcal{H} \subset \mathcal{F}(T)$ the $\mathcal{H}$-dual problem

$$(CID_{\mathcal{H}}) \qquad \sup_{H \in \mathcal{H},\ \lambda \in \mathbb{R}_+^H} \inf_{x \in X} \left\{ f(x) + \sum_{t \in H} \lambda_t g_t(x) \right\}$$

Relaxed convex infinite Farkas-type lemmas

- Consider again the convex infinite problem of the previous section:

$$(CIP) \quad \min f(x) \ \text{ s.t. } \ x \in A =: \{x \in X : g_t(x) \leq 0, t \in T\}$$

- In Dinh-G-López-Volle (2021a,b) we associate with a given family $\emptyset \neq \mathcal{H} \subset \mathcal{F}(T)$ the $\mathcal{H}$-dual problem

$$(CID_{\mathcal{H}}) \qquad \sup_{H \in \mathcal{H},\ \lambda \in \mathbb{R}_+^H} \inf_{x \in X} \left\{ f(x) + \sum_{t \in H} \lambda_t g_t(x) \right\}$$

- Taking $\mathcal{H} = \mathcal{F}(T)$ one gets the usual Lagrangian-Haar dual in CIP (el problema (CID) de la sección anterior).

- Consider again the convex infinite problem of the previous section:

$$(CIP) \quad \min f(x) \text{ s.t. } x \in A =: \{x \in X : g_t(x) \leq 0, t \in T\}$$

- In Dinh-G-López-Volle (2021a,b) we associate with a given family $\emptyset \neq \mathcal{H} \subset \mathcal{F}(T)$ the $\mathcal{H}$-dual problem

$$(CID_{\mathcal{H}}) \qquad \sup_{H \in \mathcal{H},\ \lambda \in \mathbb{R}_+^H} \inf_{x \in X} \left\{ f(x) + \sum_{t \in H} \lambda_t g_t(x) \right\}$$

- Taking $\mathcal{H} = \mathcal{F}(T)$ one gets the usual Lagrangian-Haar dual in CIP (el problema (CID) de la sección anterior).
- Other interesting families are $\mathcal{H}_1 := \{\{t\},\ t \in T\}$ and, for $T = \mathbb{N}$, $\mathcal{H}_{\mathbb{N}} := \{\{1, \ldots, n\},\ n \in \mathbb{N}\}$.

Relaxed convex infinite Farkas-type lemmas

- (CIP) (respectively, $(CID_{\mathcal{H}})$) is *$\mathcal{H}$-reducible* if there exists $H \in \mathcal{H}$ such that $\inf(CIP) = \inf(P_H)$ (resp. $\sup(CID_{\mathcal{H}}) = \sup(D_H)$), where

$$(P_H) \quad \min f(x) \text{ s.t. } g_t(x) \leq 0, t \in H$$

and

$$(D_H) \qquad \sup_{\lambda \in \mathbb{R}_+^H} \inf_{x \in X} \left\{ f(x) + \sum_{t \in H} \lambda_t g_t(x) \right\}$$

Relaxed convex infinite Farkas-type lemmas

- (CIP) (respectively, $(CID_{\mathcal{H}})$) is *$\mathcal{H}$-reducible* if there exists $H \in \mathcal{H}$ such that $\inf(CIP) = \inf(P_H)$ (resp. $\sup(CID_{\mathcal{H}}) = \sup(D_H)$), where

$$(P_H) \quad \min f(x) \text{ s.t. } g_t(x) \leq 0, t \in H$$

and

$$(D_H) \qquad \sup_{\lambda \in \mathbb{R}_+^H} \inf_{x \in X} \left\{ f(x) + \sum_{t \in H} \lambda_t g_t(x) \right\}$$

- $\mathcal{F}(T)$-reducibility of (CIP) is the natural extension of LSIP reducibility.

- (CIP) (respectively, $(CID_{\mathcal{H}})$) is *$\mathcal{H}$-reducible* if there exists $H \in \mathcal{H}$ such that $\inf(CIP) = \inf(P_H)$ (resp. $\sup(CID_{\mathcal{H}}) = \sup(D_H)$), where

$$(P_H) \quad \min f(x) \text{ s.t. } g_t(x) \le 0, t \in H$$

and

$$(D_H) \qquad \sup_{\lambda \in \mathbb{R}_+^H} \inf_{x \in X} \left\{ f(x) + \sum_{t \in H} \lambda_t g_t(x) \right\}$$

- $\mathcal{F}(T)$-reducibility of (CIP) is the natural extension of LSIP reducibility.
- The next result in Dinh-G-López-Volle (2021a) characterizing the $\mathcal{H}$-reducibility of (CIP) and $(CID_{\mathcal{H}})$ involves the set

$$\mathcal{A}_{\mathcal{H}} := \bigcup_{H \in \mathcal{H}, \lambda \in \mathbb{R}_+^H} \operatorname{epi} \left(f + \sum_{t \in H} \lambda_t f_t \right)^* \subset X^* \times \mathbb{R}$$

Relaxed convex infinite Farkas-type lemmas

- It also involves the strong duality between (CIP) and $(CID_{\mathcal{H}})$, as well as the following property:

$$(\mathrm{R}_{\mathcal{H}}) \quad \inf(P_H) = \max(D_H), \forall H \in \mathcal{H}$$

- It also involves the strong duality between (CIP) and $(CID_{\mathcal{H}})$, as well as the following property:

$$(\mathrm{R}_{\mathcal{H}}) \quad \inf(P_H) = \max(D_H), \forall H \in \mathcal{H}$$

- Property $(\mathrm{R}_{\mathcal{H}})$ is, e.g., satisfied in the two well-known situations below:
  1. (CIP) is an LSIP problem.
  2. (CIP) satisfies

$$\forall H \in \mathcal{H},\ \exists a_H \in \operatorname{dom} f \text{ such that } g_t(a_H) < 0,\ \forall t \in H.$$

  This is in particular satisfied under the (generalized) Slater condition; i.e.,

$$\exists a \in \operatorname{dom} f \text{ such that } g_t(a) < 0 \text{ for all } t \in T.$$

Theorem (CIP $\mathcal{H}$-reducibility)

Let $\alpha := \inf(CIP) \in \mathbb{R}$ *and consider the following assertions:*
(i) $(0_{X^*}, -\alpha) \in \mathcal{A}_{\mathcal{H}}$.
(ii) $\mathcal{H}$*-strong duality holds (i.e.,* $\alpha = \max(CID_{\mathcal{H}})$*).*
(iii) $(CID_{\mathcal{H}})$ *is* $\mathcal{H}$*-reducible and* $\alpha = \sup(CID_{\mathcal{H}})$*.*
(iv) (CIP) *is* $\mathcal{H}$*-reducible.*
Then, one has (i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv).
Moreover, if $(\mathrm{R}_{\mathcal{H}})$ *holds, then* (i), (ii), (iii), *and* (iv) *are equivalent.*

Corollary (LIP $\mathcal{H}$-reducibility)

Assume that $\alpha := \inf(LIP) \in \mathbb{R}$ *and*

$$\forall H \in \mathcal{H}, \exists a_H \in X : \; \langle a_t^*, a_H \rangle < b_t, \; \forall t \in H$$

TFA:

(i) $-(x^*, \alpha) \in \bigcup_{H \in \mathcal{H}} \operatorname{cone}\left(\{(a_t^*, b_t), \; t \in H\} + \{0_{X^*}\} \times \mathbb{R}_+\right).$

(ii) $\mathcal{H}$*-strong duality holds (i.e.,* $\alpha = \max(LID_{\mathcal{H}})$*).*

(iii) $(LID_{\mathcal{H}})$ *is* $\mathcal{H}$*-reducible and zero* $\mathcal{H}$*-duality holds (entailing* $\alpha = \sup(LID_{\mathcal{H}})$*).*

(iv) (LIP) *is* $\mathcal{H}$*-reducible.*

The union in (i) is the characteristic cone of (LIP) whenever the family $\mathcal{H}$ is directed (i.e., $\forall H, K \in \mathcal{H} \; \exists L \in \mathcal{H} : H \cup K \subset L$) and covering (i.e., $\bigcup_{H \in \mathcal{H}} H = T$), e.g., when $\mathcal{H} = \mathcal{F}(T)$.

A subset $\mathcal{A} \subset X^* \times \mathbb{R}$ is said to be w^*-closed regarding another subset $\mathcal{B} \subset X^* \times \mathbb{R}$ if $(\operatorname{cl} \mathcal{A}) \cap \mathcal{B} = \mathcal{A} \cap \mathcal{B}$.

Theorem ($\mathcal{H}$-strong duality)

Let $\mathcal{H} \subset \mathcal{F}(T)$ be a covering family and consider the following statements:
(i) $\mathcal{H}$-strong duality holds.
(ii) $\mathcal{A}_{\mathcal{H}}$ is w^*-closed convex regarding $\{0_{X^*}\} \times \mathbb{R}$.
Then we have (i) $\Rightarrow$ (ii).
If, moreover, $\{f;\ g_t, t \in T\} \subset \Gamma(X)$ and $A \cap \operatorname{dom} f \neq \emptyset$, then (i) $\Leftrightarrow$ (ii).

Relaxed convex infinite Farkas-type lemmas

The next results are from Dinh-G-López-Volle (2021b).

Theorem (Characterization of $\mathcal{H}$-Farkas lemma)

Let $\mathcal{H} \subset \mathcal{F}(T)$ be a covering family. Assume that $\{f;\ g_t, t \in T\} \subset \Gamma(X)$, $A \cap (\mathrm{dom}\, f) \neq \emptyset$, and consider the following statements:
(I) $\mathcal{A}_{\mathcal{H}}$ *is w^*-closed convex regarding* $\{0_{X^*}\} \times \mathbb{R}$.
(II) *For* $\alpha \in \mathbb{R}$, *TFA:*

$$\begin{array}{ll} \mathrm{(i)} & [g_t(x) \leq 0, \forall t \in T] \Longrightarrow f(x) \geq \alpha, \\ \mathrm{(ii)} & \exists H \in \mathcal{H} \text{ and } \lambda \in \mathbb{R}_+^H : f(x) + \sum_{t \in H} \lambda_t g_t(x) \geq \alpha, \forall x \in X. \end{array}$$

Then, $[\mathrm{(I)} \Longrightarrow \mathrm{(II)}]$, *and the converse implication,* $[\mathrm{(II)} \Longrightarrow \mathrm{(I)}]$, *holds when* $\inf(CIP) \in \mathbb{R}$.

Theorem (Primal-dual $\mathcal{H}-$optimality condition)

Let $\bar{x} \in A \cap (\operatorname{dom} f)$, $H \in \mathcal{H}$ *and* $\lambda \in \mathbb{R}_+^H$. *TFA:*

(i) $\bar{x} \in \operatorname{sol}(CIP)$, $(H, \lambda) \in \operatorname{sol}(CID_{\mathcal{H}})$, *and*

$$\inf(CIP) = \sup(CID_{\mathcal{H}})$$

(ii) $f(\bar{x}) = \inf_X \left(f + \sum_{t \in H} \lambda_t g_t \right)$, *and* $\lambda_t g_t(\bar{x}) = 0, \ \forall t \in H$.

(iii) $0_{X^*} \in \partial \left(f + \sum_{t \in H} \lambda_t g_t \right)(\bar{x})$, *and* $\lambda_t g_t(\bar{x}) = 0, \ \forall t \in H$.

Corollary (1st primal $\mathcal{H}-$optimality condition)

Assume that $\inf(CIP) = \max(CID_{\mathcal{H}})$ *and let* $\bar{x} \in A \cap (\operatorname{dom} f)$*.*
TFA:
(i) $\bar{x} \in \operatorname{sol}(CIP)$.
(ii) $\forall (H, \lambda) \in \operatorname{sol}(CID_{\mathcal{H}})$, *we have*

$$0_{X^*} \in \partial \left(f + \sum_{t \in H} \lambda_t g_t \right) (\overline{x}), \text{ and } \lambda_t g_t(\bar{x}) = 0, \ \forall t \in H. \qquad (2)$$

(iii) $\exists (H, \lambda) \in \operatorname{sol}(CID_{\mathcal{H}})$ *such that (2) is fulfilled.*

Corollary (2nd primal $\mathcal{H}-$optimality condition)

Let $\mathcal{H} \subset \mathcal{F}(T)$ be a covering family. Assume that $\{f;\ g_t, t \in T\} \subset \Gamma(X)$ and $A \cap (\operatorname{dom} f) \neq \emptyset$. Assume further that $\mathcal{A}_{\mathcal{H}}$ is w^*-closed convex regarding $\{0_{X^*}\} \times \mathbb{R}$.
Then $\bar{x} \in \operatorname{sol}(CIP)$ iff $\exists H \in \mathcal{H}$ and $\lambda \in \mathbb{R}_+^H$ such that

$$0_{X^*} \in \partial \left(f + \sum_{t \in H} \lambda_t g_t \right) (\overline{x}), \text{ and } \lambda_t g_t(\bar{x}) = 0,\ \forall t \in H.$$

Role of $\mathcal{H}-$Farkas' lemma in the proofs
(Dinh-G-López-Volle, 2021a,b)

$$
\begin{array}{ccccc}
 & & \mathcal{H}-\text{Strong duality} & \Longrightarrow & \mathcal{H}-\text{reducibility} \\
 & & \Downarrow & & \\
 & & \mathcal{H}-\text{Farkas lemma} & & \\
 & & \Downarrow & & \\
\text{Corol. 1} & \Longleftarrow & \begin{array}{c}\text{Primal} - \text{dual} \\ \mathcal{H}-\text{optim. cond.}\end{array} & \Longrightarrow & \text{Corol. 2}
\end{array}
$$

Vector infinite Farkas-type lemmas

Vector infinite Farkas-type lemmas

- The last stop in this personal tour through Farkas-type lemmas will consist in getting rid of the following assumptions:

Vector infinite Farkas-type lemmas

- The last stop in this personal tour through Farkas-type lemmas will consist in getting rid of the following assumptions:

1. Scalar objective function.

Vector infinite Farkas-type lemmas

- The last stop in this personal tour through Farkas-type lemmas will consist in getting rid of the following assumptions:

1. Scalar objective function.
2. Convexity of that function.

- The last stop in this personal tour through Farkas-type lemmas will consist in getting rid of the following assumptions:

1. Scalar objective function.
2. Convexity of that function.
3. Convexity and closedness of the feasible set.

Vector infinite Farkas-type lemmas

- The last stop in this personal tour through Farkas-type lemmas will consist in getting rid of the following assumptions:

1. Scalar objective function.
2. Convexity of that function.
3. Convexity and closedness of the feasible set.

- We now motivate the optimization model considered here with a simple example where $\|.\|_i$ stands for the L_i norm, $i \in \mathbb{N}$, and $\|.\|_\infty$ is the uniform (or Chebyshev) norm on the space Z of bounded real-valued functions on a compact interval $T \subset \mathbb{R}$ which are continuous a.e.

An example (1)

- A decision maker (DM) wants to find a best approximation of a given function $h \in Z$ from above by polynomials.

An example (1)

- A decision maker (DM) wants to find a best approximation of a given function $h \in Z$ from above by polynomials.
- Identifying $x = (x_k)_{k=1}^{\infty} \in \mathbb{R}^{(\mathbb{N})}$ with the polynomial $p(x)(t) = \sum_{k \in \mathbb{N}} x_k t^{k-1}$, the decision space is $X = \mathbb{R}^{(\mathbb{N})}$.

An example (1)

- A decision maker (DM) wants to find a best approximation of a given function $h \in Z$ from above by polynomials.
- Identifying $x = (x_k)_{k=1}^{\infty} \in \mathbb{R}^{(\mathbb{N})}$ with the polynomial $p(x)(t) = \sum_{k \in \mathbb{N}} x_k t^{k-1}$, the decision space is $X = \mathbb{R}^{(\mathbb{N})}$.
- The side constraints can be written as

$$\sum_{k=1}^{n} x_k t^{k-1} \geq h(t), \forall t \in T,$$

or, equivalently,

$$G(x) \in -S,$$

with $G(x) := h - p(x) \in Z$ and $S := Z_+$.

An example (2)

- Since $h - p(x) \in Z$, $f_i(x) := \|h - p(x)\|_i$ is well-defined $\forall x \in X$, $\forall i = 1, ..., \infty$.

An example (2)

- Since $h - p(x) \in Z$, $f_i(x) := \|h - p(x)\|_i$ is well-defined $\forall x \in X$, $\forall i = 1, ..., \infty$.
- The best uniform approximation from above to h is

$$(IP) \quad \min \ \{f_\infty(x) : G(x) \in -S\}$$

An example (2)

- Since $h-p(x)\in Z$, $f_i(x):=\left\|h-p(x)\right\|_i$ is well-defined $\forall x\in X$, $\forall i=1,...,\infty$.
- The best uniform approximation from above to h is

$$(IP)\quad \min\ \{f_\infty(x):G(x)\in -S\}$$

- The DM could also balance the L_∞ and the L_1 norms by solving the biobjective problem

$$(BOIP)\quad \mathrm{Wmin}\,\{F(x):G(x)\in -S\},$$

where, $F=(f_1,f_\infty):X\longrightarrow Y:=\mathbb{R}^2$, with Y equipped with the partial ordering $\leqq_K$ induced by the cone $K:=\mathbb{R}^2_+$.

An example (2)

- Since $h - p(x) \in Z$, $f_i(x) := \|h - p(x)\|_i$ is well-defined $\forall x \in X$, $\forall i = 1, ..., \infty$.
- The best uniform approximation from above to h is

$$(IP) \quad \min \ \{f_\infty(x) : G(x) \in -S\}$$

- The DM could also balance the L_∞ and the L_1 norms by solving the biobjective problem

$$(BOIP) \quad \mathrm{Wmin}\,\{F(x) : G(x) \in -S\},$$

where, $F = (f_1, f_\infty) : X \longrightarrow Y := \mathbb{R}^2$, with Y equipped with the partial ordering $\leqq_K$ induced by the cone $K := \mathbb{R}^2_+$.
- Recall that $\overline{x} \in X$ is a *weak efficient solution* of $(BOIP)$ if

$$\nexists x \in G(x) \in -S : F(x) < F(\overline{x})$$

("$<$" componentwise), i.e., if

$$F(x) \notin F(\overline{x}) - \mathrm{int}\,\mathbb{R}^2_+, \ \forall x \in A := G^{-1}(-S)$$

An example (3)

- More generally, the DM could consider the best approximation to h from above relative to all norms $\{\|\cdot\|_i, i \in \mathbb{N} \cup \{\infty\}\}$ by solving the vector infinite program

$$(VIP): \quad \mathrm{Wmin}\{F(x) : G(x) \in -S\},$$

where $F : X \longrightarrow Y := \mathbb{R}^{\mathbb{N}\cup\{\infty\}}$ is the vector function whose $i-$th projection is $\|h - p(x)\|_i$.

An example (3)

- More generally, the DM could consider the best approximation to h from above relative to all norms $\{\|\cdot\|_i, i \in \mathbb{N} \cup \{\infty\}\}$ by solving the vector infinite program

$$(VIP): \ \ \mathrm{Wmin}\,\{F(x) : G(x) \in -S\},$$

where $F : X \longrightarrow Y := \mathbb{R}^{\mathbb{N}\cup\{\infty\}}$ is the vector function whose $i-$th projection is $\|h - p(x)\|_i$.

- Now the *space of criteria* is Y equipped with the partial ordering $\leqq_K$induced by the cone $K := \mathbb{R}_+^{\mathbb{N}\cup\{\infty\}}$, and $\overline{x} \in X$ is a *weak efficient solution* if $\nexists x \in A = G^{-1}(-S)$:

$$F(x) \notin F(\overline{x}) - \mathrm{int}\, K, \ \forall x \in A,$$

i.e.,

$$A \subset B := \{x \in X : F(x) \notin F(\overline{x}) - \mathrm{int}\, K\} \qquad (3)$$

- Consider the *vector infinite optimization* problem

$$(VIP) \qquad \mathrm{Wmin}\left\{F(x) : G(x) \in -S\right\}$$

where:

Vector infinite Farkas-type lemmas

- Consider the *vector infinite optimization* problem

$$(VIP) \qquad \mathrm{Wmin}\left\{F(x) : G\left(x\right) \in -S\right\}$$

where:

- X, Y, Z are lcHtvs.

Vector infinite Farkas-type lemmas

- Consider the *vector infinite optimization* problem

$$(VIP) \qquad \operatorname{Wmin}\{F(x) : G(x) \in -S\}$$

where:

- X, Y, Z are lcHtvs.
- Y is equipped with the partial ordering $\leqq_K$ induced by a nonempty, closed, pointed, convex cone K such that $\operatorname{int} K \neq \emptyset$; this ordering is extended to $Y^\bullet := Y \cup \{\pm\infty_Y\}$.

Vector infinite Farkas-type lemmas

- Consider the *vector infinite optimization* problem

$$(VIP) \qquad \mathrm{Wmin}\left\{F(x) : G(x) \in -S\right\}$$

where:

- X, Y, Z are lcHtvs.
- Y is equipped with the partial ordering $\leqq_K$ induced by a nonempty, closed, pointed, convex cone K such that $\operatorname{int} K \neq \emptyset$; this ordering is extended to $Y^\bullet := Y \cup \{\pm\infty_Y\}$.
- Similarly, Z is another lcHtvs equipped with the partial ordering $\leqq_S$ induced by a convex cone $S \neq \emptyset$, which is extended to $Z^\bullet := Z \cup \{\pm\infty_Z\}$.

- Consider the *vector infinite optimization* problem

$$(VIP) \qquad \mathrm{Wmin}\left\{F(x) : G(x) \in -S\right\}$$

where:

- X, Y, Z are lcHtvs.
- Y is equipped with the partial ordering $\leqq_K$ induced by a nonempty, closed, pointed, convex cone K such that $\operatorname{int} K \neq \emptyset$; this ordering is extended to $Y^\bullet := Y \cup \{\pm\infty_Y\}$.
- Similarly, Z is another lcHtvs equipped with the partial ordering $\leqq_S$ induced by a convex cone $S \neq \emptyset$, which is extended to $Z^\bullet := Z \cup \{\pm\infty_Z\}$.
- $F : X \to Y^\bullet$ and $G : X \to Z^\bullet$.

Vector infinite Farkas-type lemmas

Different vector Farkas-type results have been given in Dinh-G-López-Mo (2019) where the key tool is the conjugate calculus with vector functions (Tanino, 1992).
In the next result, $0_{\mathcal{L}}$ is the null element of the space $\mathcal{L}(X, Y)$ of linear continuous mappings from X to Y while the other symbols represent extensions of homonymous concepts from the scalar to the vector setting.

Theorem (An abstract vector Farkas lemma)

Let $\bar{x} \in A \cap \operatorname{dom} F$. *TFA:*
(i) $G^{-1}(-S) \subset \{x \in X : F(x) \notin F(\bar{x}) - \operatorname{int} K\}$.
(ii) $0_{\mathcal{L}} \in \partial(F + I_A)(\bar{x})$.
(iii) $(0_{\mathcal{L}}, -F(\bar{x})) \in \operatorname{epi}_K (F + I_A)^*$.

Recall that (i) is nothing else than optimality of $\bar{x}$.

Vector infinite Farkas-type lemmas

- Finally, the characterization of $\operatorname{epi}_K (F + I_A)$ in Dinh-G-Long-López (2017) provides Farkas lemmas involving F and G (the data) instead of F and A, paying for it the price of aggregating some assumptions.

- Finally, the characterization of $\operatorname{epi}_K(F + I_A)$ in Dinh-G-Long-López (2017) provides Farkas lemmas involving F and G (the data) instead of F and A, paying for it the price of aggregating some assumptions.
- In fact, under suitable assumptions, one has

$$\operatorname{epi}_K(F + I_A)^* = \operatorname{cl}\left(\bigcup_{T\in\mathcal{L}_+(S,K)} \operatorname{epi}_K(F + T\circ G)^*\right),$$

or even

$$\operatorname{epi}_K(F + I_A)^* = \bigcup_{T\in\mathcal{L}_+(S,K)} \operatorname{epi}_K(F + T\circ G)^*,$$

where

$$\mathcal{L}_+(S,K) := \{T \in \mathcal{L}(Z,Y) : \; T(S) \subset K\}$$

denotes the *cone of positive operators*.

References

- Aragón-G-López-Rodriguez, Nonlinear Optimization, Springer, 2019.
- Burachik-Jeyakumar, Dual condition for the convex subdifferential sum formula with applications, *J. Convex Analysis* 12 (2005) 279-290.
- Chu, Generalization of some fundamental theorems on linear inequalities [in Chinese]. *Acta Math. Sinica* 16 (1966) 25-40.
- Dinh-G-Long-López (2019), New Farkas-type results for vector-valued functions: a non-abstract approach, *J. Optim. Theory Appl.* 182 (2019) 4-29.
- Dinh-G-López, From linear to convex systems: consistency, Farkas' lemma and applications, *J. Convex Analysis* 13 (2006) 279-290.
- Dinh-G.-López-Mo, Farkas-type results for vector-valued functions with applications, *J. Optim. Theory Appl.* 173 (2017) 357-390.

Dinh-G-López-Son, New Farkas-type constraint qualifications in convex infinite programming, *ESAIM: COCV* 13 (2007) 580-597.

Dinh-G-López-Volle, Convex inequalities without constraint qualification nor closedness condition, and their applications in optimization. *Set-Valued Var. Anal.* 18 (2010) 423-445.

Dinh-G-López-Volle, Characterizations of robust and stable duality for linearly perturbed uncertain optimization problems, in From Analysis to Visualization [hommage to J.M. Borwein], pp. 43-74, Springer (2020)

Dinh-G-López-Volle, Relaxed Lagrangian and convex infinite optimization duality: reducibility and strong duality, submitted to *Optimization* (2021a), http://arxiv.org/abs/2106.01662.

Dinh-G-López-Volle, Relaxed Lagrangian duality in convex infinite optimization: reverse strong duality and optimality, *J. Appl. Numer. Optim.*, to appear (2021b), http://arxiv.org/abs/2106.09299.

- Dinh-Mordukhovich-Nghia, Subdsiierentials of value functions and optimality conditions for DC and bilevel infinite and semi-infinite programs, *Math. Programming* 123 (2010) 101-138.
- Dinh-Vallet-Nghia, Farkas-type results and duality for DC programs with convex constraints, *J. Convex Anal.* 15 (2008) 235-262.
- Doagooei-Mohebi, Dual characterizations of the set containments with strict cone-convex inequalities in Banach spaces, *J. Global Optim.* 43 (2009) 577-591.
- Elliott-Kopp, Mathematics of Financial Markets, Springer, 2005.
- Fajardo-G-Rodríguez-Vicente, Even Convexity and Optimization: Handling Strict Inequalities, Springer, 2020.
- Fang-Li-Ng, Constraint Qualifications for Extended Farkas's lemma and Lagrangian Dualities in Convex Infinite Programming, *SIAM J. Optim.* 20 (2009) 1311-1332.

Fang-Li-Ng, Constraint qualifications for optimality conditions and total Lagrange dualities in convex infinite programming, *Nonlinear Anal.* 73 (2010) 1143-1159.

Fang-Zhao, Optimality conditions for convex and DC infinite optimization problems, *J. Nonlinear Convex Anal.* 17 (2016) 683–700.

Franklin, Mathematical methods of economics, *Amer. Math. Monthly* 90 (1983) 229-244.

Gale-Kuhn-Tucker, Linear programming and the theory of games. In: *Activity Analysis of Production and Allocation*, pp. 317-329, Wiley, 1951.G-Jeyakumar-Dinh, Dual characterizations of set containments with strict convex inequalities. *J. Global Optim.* 34 (2006) 33-54.

G-González-Martinez_Legaz-Todorov, Motzkin decomposition of closed convex sets, *J. Math. Anal. Appl.* 364 (2010) 209-221.

G-Iusem-Martinez_Legaz-Todorov, Motzkin decomposition of

- G-Jornet-Puente, Linear Optimization: Theory, Methods and Models (Spanish), McGraw-Hill, 2004.
- G-Jornet-Puente-Todorov: Analytical linear inequality systems and optimization, *J. Optim. Theory Appl.* 103 (1999) 95-119.
- G-López, *Linear Semi-infinite Optimization*, Wiley, 1998.
- G-López, Recent contributions to linear semi-infinite optimization, *4OR* 15 (2017) 221-264.
- G-López-Pastor, J. Farkas-Minkowski systems in semi-infinite programming, *Appl. Math. Optim.* 7 (1981) 295-308.
- G-López-Volle, Primal attainment in convex infinite optimization duality, J. Conv. Anal. 21 (2014) 1043-1064.

- G-Martinez_Legaz-Todorov, On Motzkin decomposable sets and functions, *J. Math. Anal. Appl.* 372 (2010) 525-537.

- Jeyakumar, Farkas' lemma: Generalizations. In *Encyclopedia of Optimization II*, pp. 87- 91, Kluwer, 2001.

- Jeyakumar, Characterizing set containments involving infinite convex constraints and reverse-onvex constraints, *SIAM J. Optim.* 13, (2003) 947-959.

- Jeyakumar-Lee-Lee, Sums of squares characterizations of containment of convex semialgebraic sets, *Pac. J. Optim.* 12 (2016) 29-42.

- Jeyakumar-Li, Farkas' lemma for separable sublinear inequalities without qualifications, *Opt. Letters* 3 (2009) 537-545.

- Kellner-Theobald-Trabandt, Containment problems for polytopes and spectrahedra, *SIAM J. Optim.* 23 (2013) 1000-1020.

- Li-Ng-Pong, Constraint qualifications for convex inequality systems with applications in constrained optimization, *SIAM J. Optim.* 19 (2008) 163-187.
- Mangasarian, Generalized support vector machines, in *Advances in Large Margin Classifiers*, MIT Press, pp. 135-146, 2000.
- Mangasarian, Set containment characterization, *J. Global Optim.* 24 (2002) 473-480.
- Prékopa, Linear Inequalities, duality theorems and their financial applications, Kassay, Proc. Int. Conf. In memoriam Gyula Farkas, Presa Universitară Clujeană, pp. 8-21, 2006.
- Robotka-Vizvari, A little theory for the control of an assembly robot using Farkas Theorem, Proc. Int. Conf. In memoriam Gyula Farkas, Presa Universitară Clujeană, pp. 50-64, 2006.
- Rodriguez-Vicente_Pérez, On finite linear systems containing strict inequalities, *J Optim. Theory Appl.* 173 (2017) 131-154.

- Rodriguez-Vicente_Pérez, Duality for sets of strong Slater points, submitted to SVVA (2021).
- Suzuki, Set containment characterization with strict and weak quasiconvex inequalities, *J. Global Optim.* 47 (2010) 273-285.
- Suzuki-Kuroiwa, Set containment characterization for quasiconvex programming, *J. Global Optim.* 45 (2009) 551-563.
- Suzuki-Kuroiwa, On set containment characterization and constraint qualification for quasiconvex programming, *J. Optim. Theory Appl.* 149 (2011) 554-563.
- Tanino, Conjugate duality in vector optimization, *J. Math. Anal. Appl.* 167 (1992) 84-97.
- Wei-Yao, On constraint qualifications of a nonconvex inequality, Optim. Lett. 12 (2018) 1117–1139.

Thank you for your attention!