Farkas’ lemma: some extensions and applications

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Gyula Farkas and Hungarian mathematics
Outline

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- The classical Farkas’ lemma
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Why so many? Because it was the inception of a new method.

"Professor H. F. Bohnenblust once told me something about research. He had supervised many successful Ph.D. thesis projects - and a few unsuccessful ones. He said this: ‘The unsuccessful projects start with some famous old problem (prove the Riemann hypothesis) and then look for a method to solve it. The successful projects start with some new method and then look for a problem’ (...). I hope to convince you that every mathematician should know the Farkas theorem [lemma for us] and should know how to use it" (Joel Franklin, 1983).
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Gyula Farkas and Hungarian mathematics

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His publications allowed him to become a professor at the University of Pest and, since 1887 on, at the new University of Kolozsáar (present-day Cluj) -created in 1872 by the Hungarian minister of Religion and Education Eötvös-, better known as ‘the Göttingen of the monarchy’ (Farkas coincided there with Haar, Riesz, Radó, etc.).
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The first contemporary mathematics competition for secondary school students was the Eötvös competition held in 1894 (at least, Haar and Radó won the 1st price, as Riesz’s brother did).
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The first contemporary mathematics competition for secondary school students was the Eötvös competition held in 1894 (at least, Haar and Radó won the 1st price, as Riesz’s brother did).

"The winners of these competitions, so to say, overlap with the set of mathematicians and physicists who later became well-respected world figures" (John von Neumann, 1929, who couldn’t compete due to WWI).
The classical Farkas’ lemma
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We start with scalar optimization problem of the form

\[
(P) \min f(x) \text{ s.t. } x \in A,
\]

where \( A \subseteq X \) is the \textit{feasible set}, \( X \) is the \textit{decision space}, and 
\( f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm \infty\} \) is the \textit{objective function}. 

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- \(\bar{x} \in A\) is a minimizer of \((P)\), \(\bar{x} \in \text{sol}(P)\) in short, when

\[x \in A \Rightarrow f(\bar{x}) \leq f(x)\]

or, equivalently, denoting \(h(x) := f(\bar{x}) - f(x)\),

\[A \subset B := [h \leq 0] := \{x \in X : h(x) \leq 0\}\]
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or, equivalently, denoting \( h(x) := f(\bar{x}) - f(x) \),

\[
A \subset B := [h \leq 0] := \{ x \in X : h(x) \leq 0 \}
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- A Farkas-type lemma is a characterization of the set containment \( A \subset B \), when \( B \) is a sublevel set of some function.
The classical Farkas’ lemma

Let $A = \{ x \in \mathbb{R}^n : \langle a_t, x \rangle \leq b_t, \ t = 1, \ldots, m \}$.

Lemma (Non-homogeneous Farkas)

If $A \neq \emptyset$ and $(a, b) \in \mathbb{R}^{n+1}$, then,

$$A \subset \{ x \in \mathbb{R}^n : \langle a, x \rangle \leq b \}$$

$$\iff \begin{pmatrix} a \\ b \end{pmatrix} \in \text{cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t = 1, \ldots, m; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\}$$

The cone above is the characteristic cone of this system. From this affine/affine Farkas lemma it is easy to prove the four pillars of LP theory.

Theorem (Existence)

$$A \neq \emptyset \iff \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \notin \text{cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t = 1, \ldots, m; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\}$$

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The classical Farkas’ lemma

We now consider the LP problem

\[(LP) \quad \min \, \langle c, x \rangle \text{ s.t. } x \in A =: \{ \langle a_t, x \rangle \leq b_t, t = 1, \ldots, m \}.\]

The set of active indices and the active cone at \( \bar{x} \in A \) are

\[T(\bar{x}) = \{ t \in T := \{1, \ldots, m\} : \langle a_t, \bar{x} \rangle = b_t \}\]

\[\text{cone} \{ a_t, t \in T(\bar{x}) \}\]

Theorem (LP Optimality Theorem)

Let \( \bar{x} \in A \). Then \( \bar{x} \in \text{sol}(LP) \) iff

\[-c \in \text{cone} \{ a_t, t \in T(\bar{x}) \}, \quad (1)\]

(geometric form of the KKT condition).
The classical Farkas’ lemma

The dual problem of \((LP)\) consists in the maximization of lower bounds for \(\{\langle c, x \rangle : x \in A\}\):

\[
(LD) : \sup_{\lambda \in \mathbb{R}_+^m} - \sum_{t=1}^{m} \lambda_t b_t
\]

s.t.
\[
\sum_{t=1}^{m} \lambda_t a_t = -c.
\]

**Theorem (LP Duality Theorem)**

*If \((LP)\) is bounded, i.e., \(\inf (LP) \in \mathbb{R}\), then \((LP)\) and \((LD)\) are both solvable and their optimal values coincide, i.e., \(\min (LP) = \max (LD)\).*
The classical Farkas’ lemma

**Theorem (Motzkin)**

Any polyhedral convex set is the sum of a polytope with a polyhedral convex cone.

**Role of Farkas’ lemma in the proofs**
(see, e.g., G-Jornet-Puente, 2004)

Existence theor. $\iff$ Farkas lemma $\implies$ Optimality theor.

$\Downarrow$

Motzkin theorem

$\Downarrow$

Duality theorem
The classical Farkas’ lemma

The next famous result was known by Karush, 1936, but it was first published by Kuhn-Tucker, 1951. Let

\[(NLP) \quad \min f(x) \text{ s.t. } g_t(x) \leq 0, \; t = 1, \ldots, m,\]

where all functions are $C^1$. Let $A$ be the feasible set of $(P)$ and

\[T(x) := \{ t \in \{1, \ldots, m\} : g_t(x) = 0 \}\]

be the set of active indices at $x \in A$.

**Theorem (NLP Optimality Theorem)**

*If* $x$ *is a local minimum of* $(NLP)$ *and* $\{\nabla g_t(x), \; t \in T(x)\}$ *is linearly independent, then* $\exists \lambda_t \geq 0, \; t \in T(x)$, *such that*

\[\nabla f(x) \in \text{cone} \{\nabla g_t(x), \; t \in T(x)\}\]

*(geometric form of the KKT condition).*
The classical Farkas’ lemma

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- Under the linear independence assumption, it can be shown that

  \[
  \{ x \in \mathbb{R}^n : \langle \nabla g_t (\bar{x}) , x \rangle \leq 0, t \in T (\bar{x}) \} 
  \subseteq \{ x \in \mathbb{R}^n : \langle \nabla f (\bar{x}) , x \rangle \geq 0 \} 
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  Then, applying the **homogeneous Farkas lemma** to the above inclusion \( A \subset B \), one gets

  \[-\nabla f (\bar{x}) \in \text{cone} \{ \nabla g_t (\bar{x}) , t \in T (\bar{x}) \}\]
A bit of history

- 1838: Ostrogradski tries to prove the NLP optimality theorem (with \( f \) being the potential function of a conservative field), but omits the constraint qualification and the characterization of the inclusion

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A := \{ x \in \mathbb{R}^n : \langle a_t, x \rangle \leq 0, t = 1, \ldots, m \} \subset [\langle a, \cdot \rangle \leq 0]
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1894: The physicist Farkas observes the 2nd omission (his future lemma).

1902: Farkas gives the 1st correct proof of the linear/linear (or homogeneous) Farkas’ lemma.

1911: Minkowski proves the affine/affine (also called non-homogeneous) Farkas’ lemma.
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Some applications of the classical Farkas’ lemma

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3 Machine learning (Mangasarian, 2000 and 2002): the incorporation of prior knowledge in the form of a polyhedral knowledge set $A$ in the construction of a linear classifier is modelled as the set containment $A \subset [\langle a, \cdot \rangle \leq b]$.


5 Economics (Franklin, 1983).

6 Finance (Elliott-Kopp, 2005; Prékopa, 2006).

7 Robotics (Robotka-Vizvari, 2006).
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The classical Farkas’ lemma

Extensions in the finite setting
(i.e., with $A$ being solution set of a system of finitely many variables and constraints)

- $A \subset B := [h \leq 0]$, only with weak inequalities: Jeyakumar (2001) and Dinh-Jeyakumar (2014) present selections of Farkas-type lemmas for conic-linear, conic-sublinear and convex inequality systems, conic-convex systems, classes of non-convex systems such as DC systems, composite convex systems and quadratic systems, and inequality systems involving uncertain vectors, matrices, and polynomials.
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- Similar, with strict inequalities (and maybe weak ones): results can be found in the source book Fajardo-G-Rodriguez-Vicente_Pérez (2020) as well as in Rodriguez-Vicente_Pérez (2021), which also contains an existence theorem for convex systems.
The classical Farkas’ lemma

- The containment problem consists in characterizing, in terms of the data, the inclusion $A \subset B$, where the contained $A$ and the container $B$ are solution sets of systems:
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- It was solved by Mangasarian (2000, 2002) for linear and convex-reverse convex quadratic systems with finitely many variables and constraints.
The classical Farkas’ lemma

Other contributions to the containment problem:
  - Jeyakumar (2003), for converse and reverse convex systems.
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- Suzuki (2010), for quasiconvex systems containing strict inequalities.
- Jeyakumar-Lee-Lee (2016), for SOS-convex polynomial systems.
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Extensions to the linear semi-infinite setting

A Farkas-type lemma involving the set

\[ A = \{ x \in X : g_t (x) \leq 0, t \in T \} \]

is called semi-infinite when \( \text{card} \ T \) or \( \text{dim} \ X \) is infinite, but not both.
The classical Farkas’ lemma

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- \( |\text{card} \ T| < \infty \) and \( \text{dim} \ X = \infty \): In 1924 Haar considered
  \[ X = C([0,1]), \langle a, x \rangle = \int_0^1 a(s)x(s) \, ds, \forall a, x \in X, \]
  and
  \[ A = \{ x \in X : \langle a_t, x \rangle \leq 0, t \in T \} \] with \( T \) finite. He proved the following linear/linear Farkas lemma: If \( a \in X \) and \( \{a_t, t \in T\} \subset X \) is linearly independent, then
  \[ A \subset \{ x \in X : \langle a, x \rangle \leq 0 \} \]
  \[ \iff a \in \text{cone} \{a_t, t \in T\} \]
The classical Farkas’ lemma

- $|\text{card } T| = \infty$ and $\dim X = n$: G-López-Pastor (1981) "rediscovered" the following basic result for LSIP theory (see G-López, 1998): if $(a, b) \in \mathbb{R}^{n+1}$ and $A := \{x \in \mathbb{R}^n : \langle a_t, x \rangle \leq b_t, t \in T\} \neq \emptyset$, then,

$$A \subset \{x \in \mathbb{R}^n : \langle a, x \rangle \leq b\} \iff \begin{pmatrix} a \\ b \end{pmatrix} \in \text{cl cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix} : t \in T; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\}$$
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  \]

- This affine/affine Farkas’ lemma was actually the LSIP version of Theorem 2 in Chu (1966), on LIP:
The classical Farkas’ lemma

Duality theorems in LSIP
(mostly proved in G-López (1998) thanks to Farkas’ lemma.)
A linear system satisfies the $FMCQ$ when its characteristic cone is closed and a set is called Motzkin decomposable if it is the sum of a compact convex set with a closed convex cone (these sets have been characterized by G, Gutiérrez, Iusem, Martínez Legaz and Todorov in 2010-2013). Let $\inf (LSIP), \sup (LSID) \in \mathbb{R}$.

- **Strong duality:** If FMCQ holds, then $\inf (LSIP) = \max (LSID)$.
The classical Farkas’ lemma

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- **Strong duality**: If FMCQ holds, then \( \inf (\text{LSIP}) = \max (\text{LSD}) \).
- **Reverse strong duality**: If \( -c \in \text{ri cone } \{a_t, t \in T\} \), then \( \min (\text{LSIP}) = \sup (\text{LSD}) \) and \( \text{sol} (\text{LSIP}) \) is the sum of a non-empty compact convex set with a linear subspace (from G-López-Volle, 2014, dealing with CIP).
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- **Perfect duality:** If FMCQ holds and either $-c \in \text{ri cone } \{a_t, t \in T\}$ or $A$ is Motzkin decomposable, then $\min (LSIP) = \max (LSID)$. 
The classical Farkas’ lemma

Application to the polynomial representation of polyhedra
We show the idea of Theorem 4.1 in G-Jornet-Puente-Todorov (1999) with the square $[0, 1]^2$. The characteristic cone of $\{0 \leq x_i \leq 1, i = 1, 2\}$ is

$$\text{cone } \{(-1, 0, 0), (1, 0, 1), (0, -1, 0), (0, 1, 1)\} = \text{cone } \gamma,$$

where $\gamma$ is the Lagrange-like interpolating curve through the 4 generators:
Convex infinite Farkas-type lemmas
Denote by $\Gamma(\mathcal{X})$ the set of proper lsc convex functions on $\mathcal{X}$ (a lCHtv). Recall that the Fenchel-Moreau conjugate of $h \in \Gamma(\mathcal{X})$,

$$h^*(u) := \sup\{\langle u, x \rangle - h(x) : x \in \text{dom } h\}$$

satisfies $h^{**} = h$, so that any element of $\Gamma(\mathcal{X})$ is the supremum of continuous affine functions.

**Theorem (convex/convex Farkas-type lemma)**

Let $f, g_t \in \Gamma(\mathcal{X})$, and $A = \{x \in \mathcal{X} : g_t(x) \leq 0, t \in T\} \neq \emptyset$. Then,

$$A \subset [f \leq 0] \Leftrightarrow \text{epi} f^* \subset \text{cl cone} \left\{ \bigcup_{t \in T} \text{epi} g_t^* \right\}$$
Convex infinite Farkas-type lemmas

As in LSIP, cone \( \bigcup_{t \in T} \text{epi} g_t^* \) is called characteristic cone. If it is \( w^* \)-closed, we say that the FMCQ holds. We proved it in Dinh-G-López (2006) by linearizing the functions \( g_t \) to apply Theorem 2 of Chu (1966)

**Corollary (Affine/affine Farkas lemma)**

Let \( a^*, a_t^* \in X^*, b_t, b \in \mathbb{R} \), such that
\[
A = \{ x \in X : \langle a_t^*, x \rangle \leq b_t, t \in T \} \neq \emptyset.
\]
Then,
\[
A \subset \{ x \in X : \langle a^*, x \rangle \leq b \}
\]
\[
\iff (a^*, b) \in \text{cl cone} \{ (a_t^*, b_t), t \in T; (0, 1) \}
\]

The characteristic cone in LIP is cone \( \{ (a_t^*, b_t), t \in T; (0, 1) \} \).
Convex infinite Farkas-type lemmas

**Application to linear infinite programming**

In Dinh-G-López-Volle (2020) we apply this corollary to the LIP problem

\[
\text{(LIP)} \quad \inf \langle c^*, x \rangle \text{ s.t. } \langle a_t^*, x \rangle \leq b_t, t \in T,
\]

whose Haar’s dual problem is

\[
\text{(LID)} \quad \sup_{S \in \mathcal{F}(T), \lambda \in \mathbb{R}_+^S} \left\{ - \sum_{t \in \text{supp} \lambda} \lambda_t b_t : - \sum_{t \in \text{supp} \lambda} \lambda_t a_t^* = c^* \right\},
\]

where

\[
\mathcal{F}(T) := \{ S \in \mathcal{P}(T) : 1 \leq |S| < +\infty \}
\]

and the support of \( \lambda \), \( \text{supp} \lambda \), is formed by the indices of non-zero coordinates of \( \lambda \).
Convex infinite Farkas-type lemmas

\[ \inf (LIP) = \max (LID) \in \mathbb{R} \implies (LIP) \text{ is reducible, i.e.,} \]
\[ \exists S \in \mathcal{F}(T) : \]
\[ \inf (LIP) = \inf \{ \langle c^*, x \rangle : \langle a_t^*, x \rangle \leq b_t, t \in S \} \]
Convex infinite Farkas-type lemmas

\[ \inf (LIP) = \max (LID) \in \mathbb{R} \iff (LIP) \text{ is reducible, i.e., } \exists S \in \mathcal{F}(T) : \]

\[ \inf (LIP) = \inf \{ \langle c^*, x \rangle : \langle a^*_t, x \rangle \leq b_t, t \in S \} \]

The converse statement holds if \( \{ \langle a^*_t, x \rangle \leq b_t, \forall t \in S \} \) satisfies FMCQ.
Convex infinite Farkas-type lemmas

- \( \inf (LIP) = \max (LID) \in \mathbb{R} \quad \implies \quad (LIP) \text{ is reducible, i.e.,} \exists S \in \mathcal{F}(T) : \\
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- The converse statement holds if \( \{ \langle a_t^*, x \rangle \leq b_t, \forall t \in S \} \) satisfies FMCQ.

- \( \inf (LIP) = \sup (LID) \in \mathbb{R} \quad \implies \quad (LIP) \text{ is discretizable, i.e.,} \exists (S_r)_{r \in \mathbb{N}} \subset \mathcal{F}(T) : \\
\inf (LIP) = \lim \inf \{ \langle c^*, x \rangle : \langle a_t^*, x \rangle \leq b_t, t \in S_r \} \)
Convex infinite Farkas-type lemmas

- \( \inf (LIP) = \max (LID) \in \mathbb{R} \implies \) (LIP) is \textit{reducible}, i.e., \( \exists S \in \mathcal{F} (T) : \)

  \[
  \inf (LIP) = \inf \{ \langle c^*, x \rangle : \langle a_t^*, x \rangle \leq b_t, \, t \in S \}
  \]

- The converse statement holds if \( \{ \langle a_t^*, x \rangle \leq b_t, \, \forall t \in S \} \) satisfies FMCQ.

- \( \inf (LIP) = \sup (LID) \in \mathbb{R} \implies \) (LIP) is \textit{discretizable}, i.e., \( \exists (S_r)_{r \in \mathbb{N}} \subset \mathcal{F} (T) : \)

  \[
  \inf (LIP) = \lim \inf_r \{ \langle c^*, x \rangle : \langle a_t^*, x \rangle \leq b_t, \, t \in S_r \}
  \]

- Both converse statements hold if \( X = \mathbb{R}^n \) (i.e., in LSIP).
From the LIP Farkas-type lemma (or from the existence theorem for linear systems), one can obtain the following existence theorem (Dinh-G-López, 2006).

**Theorem (Existence theorem for convex systems)**

Let \( A = \{ x \in X : g_t(x) \leq 0, t \in T \} \), with \( g_t \in \Gamma(X), \forall t \in T \). Then,

\[
A \neq \emptyset \iff (0, -1) \notin \text{cl cone} \left\{ \bigcup_{t \in T} \text{epi} g_t^* \right\}
\]
Consider now the convex infinite problem

\[(CIP) \quad \min f(x) \text{ s.t. } x \in A =: \{x \in X : g_t(x) \leq 0, t \in T\}\]
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Set constraints of the form \(x \in C\), where \(\emptyset \neq C \subset X\) is closed and convex, can be aggregated to \((CIP)\) through the indicator function \(i_C \in \Gamma(X)\).
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Set constraints of the form \(x \in C\), where \(\emptyset \neq C \subset X\) is closed and convex, can be aggregated to \((CIP)\) through the indicator function \(i_C \in \Gamma(X)\).

\([\bar{x} \in \text{sol}(CIP) \iff A \subset [f(\bar{x}) - f \leq 0]\)
Consider now the convex infinite problem

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Set constraints of the form \(x \in C\), where \(\emptyset \neq C \subset X\) is closed and convex, can be aggregated to \((CIP)\) through the indicator function \(i_C \in \Gamma(X)\).

\(\bar{x} \in \text{sol}(CIP)\) iff

\[A \subset [f(\bar{x}) - f \leq 0]\]

Since \(f(\bar{x}) - f\) is a concave function, the optimality theorem requires a convex/reverse-convex Farkas’ lemma.
The next two Farkas-type lemmas were proved in Dinh-G-López-Son (2007).

**Theorem (Convex/reverse convex Farkas’ lemma)**

Let \( f, g_t \in \Gamma(X) \), and \( A = \{ x \in X : g_t(x) \leq 0, t \in T \} \neq \emptyset \). If FMCQ holds, then,

\[
A \subset [f \geq 0] \iff 0_{X^* \times \mathbb{R}} \in \text{cl} \left( \text{epi} f^* + \text{cone} \left\{ \bigcup_{t \in T} \text{epi} g_t^* \right\} \right)
\]
The following closedness data qualification (CDQ) was introduced by Burachik-Jeyakumar (2005):

\[ \text{epi} f^* + \text{cl cone} \left\{ \bigcup_{t \in T} \text{epi} g_t^* \right\} \text{ is } w^*-\text{closed} \]
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Each of the following conditions implies CDQ:
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1. \( \text{epi} f^* + \text{cone} \left\{ \bigcup_{t \in T} \text{epi} g^*_t \right\} \text{ is } \text{w}^*\text{-closed} \).
Convex infinite Farkas-type lemmas

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1. \( \text{epi} f^* + \text{cone} \left\{ \bigcup_{t \in T} \text{epi} g_t^* \right\} \text{ is } w^*-\text{closed} \).
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Convex infinite Farkas-type lemmas

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  1. \( \text{epif}^* + \text{cone} \left\{ \bigcup_{t \in T} \text{epig}_t^* \right\} \text{ is w}^*\text{-closed.} \)
  2. FMCQ holds and \( f \) is linear.
  3. FMCQ holds and \( f \) is continuous at some point of \( A \).
Convex infinite Farkas-type lemmas

**Corollary (Convex/reverse convex Farkas' lemma)**

Let $f, g_t \in \Gamma(X)$, and $A = \{ x \in X : g_t(x) \leq 0, t \in T \} \neq \emptyset$. If CDQ holds, then,

$$A \subset [f \geq 0] \iff 0_{X^* \times \mathbb{R}} \in \text{epi} f^* + \text{cone} \left\{ \bigcup_{t \in T} \text{epi} g_t^* \right\}$$

Convex infinite Farkas-type lemmas

The elimination of the "cl" operator in the latter Farkas lemma allowed us to get optimality conditions for CIP and a saddle point theorem (in Dinh-G-López-Son, 2007). Here, \( \mathbb{R}^{(T)}_+ \) denotes the positive cone in the space \( \mathbb{R}^{(T)} \) of functions \( \lambda : T \rightarrow \mathbb{R} \) with finite support \( \text{supp} \lambda \).

**Theorem (CIP optimality theorem)**

Assume that FMCQ and CDQ hold, and let \( \bar{x} \in A \cap \text{dom} f \). Then \( \bar{x} \in \text{sol}(CIP) \) iff \( \exists \lambda \in \mathbb{R}^{(T)}_+ \) such that

\[
\partial g_t(\bar{x}) \neq \emptyset, \quad \forall t : \lambda_t > 0,
\]

\[
0_{\chi^*} \in \partial f(\bar{x}) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x})
\]

\[
\lambda_t g_t(\bar{x}) = 0, \quad \forall t \in T
\]
Convex infinite Farkas-type lemmas

The **Haar-Lagrange function** of \((CIP)\) is

\[
L(x, \lambda) := \begin{cases} 
  f(x) + \sum_{t \in T} \lambda_t f_t(x), & \text{if } \lambda \in \mathbb{R}^{(T)}_+, \\
  +\infty, & \text{otherwise}.
\end{cases}
\]

So, the **dual problem** of \((CIP)\) is

\[
(CID) \quad \sup \inf_{x \in X} L(x, \lambda), \text{ s.t. } \lambda \in \mathbb{R}_+^{(T)}
\]

**Theorem (Saddle point)**

Assume that FMCQ and CDQ hold. Then a point \(\bar{x} \in \text{sol}(CIP)\) iff \\
\(\exists \bar{\lambda} \in \mathbb{R}_+^{(T)}\) such that \((\bar{x}, \bar{\lambda})\) is a **saddle point** of \(L\), i.e.,

\[
L(\bar{x}, \lambda) \leq L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda}), \quad \forall \lambda \in \mathbb{R}_+^{(T)}
\]

In this case, \(\bar{\lambda} \in \text{sol}(CID)\).
**Theorem (Strong duality)**

If \( \inf(CIP) \in \mathbb{R} \) and FMCQ and CDQ hold, then

\[
\inf(CIP) = \max(CID)
\]

**Role of Farkas’ lemma in the proofs**


<table>
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<tr>
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Relaxed convex infinite Farkas-type lemmas
Consider again the convex infinite problem of the previous section:

\[(CIP) \quad \min f(x) \text{ s.t. } x \in A =: \{x \in X : g_t(x) \leq 0, t \in T\}\]
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\]

In Dinh-G-López-Volle (2021a,b) we associate with a given family \( \emptyset \neq \mathcal{H} \subset \mathcal{F}(T) \) the \( \mathcal{H} \)-dual problem

\[
(CID_{\mathcal{H}}) \quad \sup_{H \in \mathcal{H}, \ \lambda \in \mathbb{R}^H_+} \inf_{x \in X} \left\{ f(x) + \sum_{t \in H} \lambda_t g_t(x) \right\}
\]
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- Taking \( \mathcal{H} = \mathcal{F}(T) \) one gets the usual Lagrangian-Haar dual in CIP (el problema \( CID \) de la sección anterior).
Relaxed convex infinite Farkas-type lemmas

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- Taking \(\mathcal{H} = \mathcal{F}(T)\) one gets the usual Lagrangian-Haar dual in CIP (el problema \((CID)\) de la sección anterior).

- Other interesting families are \(\mathcal{H}_1 := \{\{t\}, \ t \in T\}\) and, for \(T = \mathbb{N}\), \(\mathcal{H}_\mathbb{N} := \{\{1, \ldots, n\}, \ n \in \mathbb{N}\}\).
Relaxed convex infinite Farkas-type lemmas

- \((CIP)\) (respectively, \((CID_H)\)) is \(\mathcal{H}\)-reducible if there exists \(H \in \mathcal{H}\) such that \(\inf(CIP) = \inf(P_H)\) (resp. \(\sup(CID_H) = \sup(D_H)\)), where

\[
(P_H) \min f(x) \text{ s.t. } g_t(x) \leq 0, \; t \in H
\]

and

\[
(D_H) \sup_{\lambda \in \mathbb{R}^H_+} \inf_{x \in X} \left\{ f(x) + \sum_{t \in H} \lambda_t g_t(x) \right\}
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\( \mathcal{F}(T) \)-reducibility of (CIP) is the natural extension of LSIP reducibility.
Relaxed convex infinite Farkas-type lemmas

- \((CIP)\) (respectively, \((CID_H)\)) is \(H\)-reducible if there exists \(H \in \mathcal{H}\) such that \(\inf(CIP) = \inf(P_H)\) (resp. \(\sup(CID_H) = \sup(D_H)\)), where

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\]

and

\[
(D_H) \quad \sup_{\lambda \in \mathbb{R}^+_H} \inf_{x \in X} \left\{ f(x) + \sum_{t \in H} \lambda_t g_t(x) \right\}
\]

- \(\mathcal{F}(T)\)-reducibility of \((CIP)\) is the natural extension of LSIP reducibility.

- The next result in Dinh-G-López-Volle (2021a) characterizing the \(H\)-reducibility of \((CIP)\) and \((CID_H)\) involves the set

\[
\mathcal{A}_H := \bigcup_{H \in \mathcal{H}, \lambda \in \mathbb{R}^+_H} \text{epi} \left( f + \sum_{t \in H} \lambda_t f_t \right)^* \subset X^* \times \mathbb{R}
\]
Relaxed convex infinite Farkas-type lemmas

- It also involves the strong duality between \((CIP)\) and \((CID_\mathcal{H})\), as well as the following property:

\[
(R_\mathcal{H}) \quad \inf(P_\mathcal{H}) = \max(D_\mathcal{H}), \forall \mathcal{H} \in \mathcal{H}
\]
It also involves the strong duality between \((CIP)\) and \((CID_H)\), as well as the following property:

\[
(R_H) \quad \inf(P_H) = \max(D_H), \forall H \in \mathcal{H}
\]

Property \((R_H)\) is, e.g., satisfied in the two well-known situations below:

1. \((CIP)\) is an LSIP problem.
2. \((CIP)\) satisfies

\[
\forall H \in \mathcal{H}, \exists a_H \in \text{dom} \ f \text{ such that } g_t(a_H) < 0, \forall t \in H.
\]

This is in particular satisfied under the \((\text{generalized})\) Slater condition; i.e.,

\[
\exists a \in \text{dom} \ f \text{ such that } g_t(a) < 0 \text{ for all } t \in T.
\]
Theorem (CIP $\mathcal{H}$-reducibility)

Let $\alpha := \inf(CIP) \in \mathbb{R}$ and consider the following assertions:

(i) $(0_{X^*}, -\alpha) \in \mathcal{A}_H$.
(ii) $\mathcal{H}$-strong duality holds (i.e., $\alpha = \max(CID_H)$).
(iii) $(CID_H)$ is $\mathcal{H}$-reducible and $\alpha = \sup(CID_H)$.
(iv) $(CIP)$ is $\mathcal{H}$-reducible.

Then, one has $(i) \iff (ii) \implies (iii) \implies (iv)$.
Moreover, if $(R_H)$ holds, then $(i), (ii), (iii)$, and $(iv)$ are equivalent.
Corollary (LIP $\mathcal{H}$-reducibility)

Assume that $\alpha := \inf(LIP) \in \mathbb{R}$ and

$$\forall H \in \mathcal{H}, \exists a_H \in X : \langle a_t^*, a_H \rangle < b_t, \:\forall t \in H$$

TFA:

(i) $- (x^*, \alpha) \in \bigcup_{H \in \mathcal{H}} \text{cone} \left( \{(a_t^*, b_t), \: t \in H\} + \{0_{x^*}\} \times \mathbb{R}_+ \right)$.

(ii) $\mathcal{H}$-strong duality holds (i.e., $\alpha = \max(LID_{\mathcal{H}})$).

(iii) $(LID_{\mathcal{H}})$ is $\mathcal{H}$-reducible and zero $\mathcal{H}$-duality holds (entailing $\alpha = \sup(LID_{\mathcal{H}})$).

(iv) $(LIP)$ is $\mathcal{H}$-reducible.

The union in (i) is the characteristic cone of $(LIP)$ whenever the family $\mathcal{H}$ is directed (i.e., $\forall H, K \in \mathcal{H} \:\exists L \in \mathcal{H} : H \cup K \subset L$) and covering (i.e., $\bigcup_{H \in \mathcal{H}} H = T$), e.g., when $\mathcal{H} = \mathcal{F}(T)$. 

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A subset $A \subset X^* \times \mathbb{R}$ is said to be $w^*$-closed regarding another subset $B \subset X^* \times \mathbb{R}$ if $(\text{cl} \ A) \cap B = A \cap B$.

**Theorem (H-strong duality)**

Let $\mathcal{H} \subset \mathcal{F}(T)$ be a covering family and consider the following statements:

(i) $\mathcal{H}$-strong duality holds.

(ii) $A_{\mathcal{H}}$ is $w^*$-closed convex regarding $\{0_{X^*}\} \times \mathbb{R}$.

Then we have (i) $\implies$ (ii).

If, moreover, $\{f; \ g_t, t \in T\} \subset \Gamma(X)$ and $A \cap \text{dom} f \neq \emptyset$, then (i) $\iff$ (ii).
The next results are from Dinh-G-López-Volle (2021b).

**Theorem (Characterization of $\mathcal{H}$-Farkas lemma)**

Let $\mathcal{H} \subset \mathcal{F}(T)$ be a covering family. Assume that 
\( \{f; g_t, t \in T\} \subset \Gamma(X), A \cap (\text{dom } f) \neq \emptyset, \) and consider the following statements:

(I) $\mathcal{A}_{\mathcal{H}}$ is $w^*$-closed convex regarding \( \{0x^*\} \times \mathbb{R}. \)

(II) For $\alpha \in \mathbb{R}$, TFA:

(i) \[ g_t(x) \leq 0, \forall t \in T \implies f(x) \geq \alpha, \]

(ii) \[ \exists H \in \mathcal{H} \text{ and } \lambda \in \mathbb{R}_+^H : f(x) + \sum_{t \in H} \lambda_t g_t(x) \geq \alpha, \forall x \in X. \]

Then, \[ (I) \implies (II), \] and the converse implication, \[ (II) \implies (I), \] holds when $\inf(CIP) \in \mathbb{R}$. 

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Theorem (Primal-dual $\mathcal{H}$—optimality condition)

Let $\bar{x} \in A \cap (\text{dom } f), H \in \mathcal{H}$ and $\lambda \in \mathbb{R}^H_+$. TFA:
(i) $\bar{x} \in \text{sol}(CIP), (H, \lambda) \in \text{sol}(CID_{\mathcal{H}})$, and
\[
\inf(CIP) = \sup(CID_{\mathcal{H}})
\]
(ii) $f(\bar{x}) = \inf_x \left( f + \sum_{t \in H} \lambda_t g_t \right)$, and $\lambda_t g_t(\bar{x}) = 0$, $\forall t \in H$.
(iii) $0_x^* \in \partial \left( f + \sum_{t \in H} \lambda_t g_t \right)(\bar{x})$, and $\lambda_t g_t(\bar{x}) = 0$, $\forall t \in H$. 
Corollary (1st primal $\mathcal{H}$—optimality condition)

Assume that $\inf(CIP) = \max(CID_{\mathcal{H}})$ and let $\bar{x} \in A \cap (\text{dom } f)$.

TFA:

(i) $\bar{x} \in \text{sol}(CIP)$.

(ii) $\forall (H, \lambda) \in \text{sol}(CID_{\mathcal{H}})$, we have

$$0 \bar{x}^* \in \partial \left( f + \sum_{t \in H} \lambda_t g_t \right) (\bar{x}), \text{ and } \lambda_t g_t(\bar{x}) = 0, \forall t \in H.$$ \hspace{1cm} (2)

(iii) $\exists (H, \lambda) \in \text{sol}(CID_{\mathcal{H}})$ such that (2) is fulfilled.
Corollary (2nd primal $\mathcal{H}$–optimality condition)

Let $\mathcal{H} \subset \mathcal{F}(T)$ be a covering family. Assume that
\[ \{ f; \ g_t, t \in T \} \subset \Gamma(X) \text{ and } A \cap (\text{dom } f) \neq \emptyset. \]
Assume further that $A_{\mathcal{H}}$ is $w^*$-closed convex regarding $\{0_{X^*}\} \times \mathbb{R}$.

Then $\bar{x} \in \text{sol}(CIP)$ iff $\exists H \in \mathcal{H}$ and $\lambda \in \mathbb{R}^H_+$ such that

\[
0_{X^*} \in \partial \left( f + \sum_{t \in H} \lambda_t g_t \right)(\bar{x}), \text{ and } \lambda_t g_t(\bar{x}) = 0, \forall t \in H.
\]
Relaxed convex infinite Farkas-type lemmas

Role of $\mathcal{H}$—Farkas’ lemma in the proofs
(Dinh-G-López-Volle, 2021a,b)

$\mathcal{H}$—Strong duality $\implies$ $\mathcal{H}$—reducibility
\[\Downarrow\]
$\mathcal{H}$—Farkas lemma
\[\Downarrow\]
Corol. 1 $\iff$ Primal $-$ dual $\mathcal{H}$—optim. cond. $\implies$ Corol. 2
Vector infinite Farkas-type lemmas
Vector infinite Farkas-type lemmas

- The last stop in this personal tour through Farkas-type lemmas will consist in getting rid of the following assumptions:

1. Scalar objective function.
2. Convexity of that function.
3. Convexity and closedness of the feasible set.

We now motivate the optimization model considered here with a simple example where $k_i$ stands for the $L_i$ norm, $i \in N$, and $k_\infty$ is the uniform (or Chebyshev) norm on the space $Z$ of bounded real-valued functions on a compact interval $T \subset \mathbb{R}$ which are continuous a.e.
The last stop in this personal tour through Farkas-type lemmas will consist in getting rid of the following assumptions:

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An example (1)

- A decision maker (DM) wants to find a best approximation of a given function $h \in Z$ from above by polynomials.
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- Identifying \( x = (x_k)_{k=1}^{\infty} \in \mathbb{R}^{(\mathbb{N})} \) with the polynomial \( p(x)(t) = \sum_{k \in \mathbb{N}} x_k t^{k-1} \), the decision space is \( X = \mathbb{R}^{(\mathbb{N})} \).
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- Identifying $x = (x_k)_{k=1}^{\infty} \in \mathbb{R}(\mathbb{N})$ with the polynomial $p(x)(t) = \sum_{k \in \mathbb{N}} x_k t^{k-1}$, the decision space is $X = \mathbb{R}(\mathbb{N})$.
- The side constraints can be written as
  \[
  \sum_{k=1}^{n} x_k t^{k-1} \geq h(t), \forall t \in T,
  \]
  or, equivalently,
  \[
  G(x) \in -S,
  \]
  with $G(x) := h - p(x) \in \mathbb{Z}$ and $S := \mathbb{Z}_+$. 

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An example (2)

- Since $h - p(x) \in Z$, $f_i(x) := \| h - p(x) \|_i$ is well-defined for all $x \in X$, $\forall i = 1, \ldots, \infty$. 

The best uniform approximation from above to $h$ is $(\text{IP})_{\min} f_{\|f\|_\infty}(x) : G(x) \in S_g$.

The DM could also balance the $L_\infty$ and the $L_1$ norms by solving the biobjective problem $(\text{BOIP})_{\min} f F(x) : G(x) \in S_g$, where, $F = (f_1, f_\infty) : X \to Y = \mathbb{R}^2$, with $Y$ equipped with the partial ordering $\leq_K$ induced by the cone $K = \mathbb{R}^2_+$. Recall that $x \in X$ is a weak efficient solution of $(\text{BOIP})_{\min} f F(x) : G(x) \in S_g$ if $\partial x \in G(x) \in S_g : F(x) < F(x)$ ("<" componentwise), i.e., if $F(x) / 2 < F(x)$ in $\mathbb{R}^2_+$, for all $x \in A = G_1(S_g)$. 

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Vector infinite Farkas-type lemmas

An example (2)

- Since $h - p(x) \in \mathbb{Z}$, $f_i(x) := \| h - p(x) \|_i$ is well-defined
  $\forall x \in X, \forall i = 1, ..., \infty$.
- The best uniform approximation from above to $h$ is

  $$(IP) \min \{ f_\infty(x) : G(x) \in -S \}$$
Vector infinite Farkas-type lemmas

An example (2)

- Since \( h - p(x) \in Z \), \( f_i(x) := \| h - p(x) \|_i \) is well-defined \( \forall x \in X, \forall i = 1, \ldots, \infty \).
- The best uniform approximation from above to \( h \) is

\[
(IP) \min \{ f_\infty(x) : G(x) \in -S \}
\]

- The DM could also balance the \( L_\infty \) and the \( L_1 \) norms by solving the biobjective problem

\[
(BOIP) \min \{ F(x) : G(x) \in -S \} ,
\]

where, \( F = (f_1, f_\infty) : X \rightarrow Y := \mathbb{R}^2 \), with \( Y \) equipped with the partial ordering \( \leq_K \) induced by the cone \( K := \mathbb{R}_+^2 \).
An example (2)

- Since $h - p(x) \in \mathbb{Z}$, $f_i(x) := \|h - p(x)\|_i$ is well-defined for all $x \in X$, $i = 1, \ldots, \infty$.

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  where, $F = (f_1, f_\infty) : X \rightarrow Y := \mathbb{R}^2$, with $Y$ equipped with the partial ordering $\leq_K$ induced by the cone $K := \mathbb{R}^2_+$. 

- Recall that $\bar{x} \in X$ is a weak efficient solution of $(BOIP)$ if

  $$\nexists x \in G(x) \in -S : F(x) < F(\bar{x})$$

  ("<" componentwise), i.e., if

  $$F(x) \notin F(\bar{x}) - \text{int} \mathbb{R}^2_+, \forall x \in A := G^{-1}(-S)$$
An example (3)

More generally, the DM could consider the best approximation to $h$ from above relative to all norms $\{\| \cdot \|_i, i \in \mathbb{N} \cup \{\infty\}\}$ by solving the vector infinite program

$$(\text{VIP}): \quad \text{Wmin} \left\{ F(x) : G(x) \in -S \right\} ,$$

where $F : X \longrightarrow Y := \mathbb{R}^{\mathbb{N} \cup \{\infty\}}$ is the vector function whose $i-$th projection is $\| h - p(x) \|_i$. 
Vector infinite Farkas-type lemmas

An example (3)

- More generally, the DM could consider the best approximation to \( h \) from above relative to all norms \( \{\| \cdot \|_i, i \in \mathbb{N} \cup \{\infty\}\} \) by solving the vector infinite program

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(VIP) : \ \text{Wmin} \left\{ F(x) : G(x) \in -S \right\},
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where \( F : X \rightarrow Y := \mathbb{R}^{\mathbb{N} \cup \{\infty\}} \) is the vector function whose \( i \)-th projection is \( \| h - p(x) \|_i \).

- Now the space of criteria is \( Y \) equipped with the partial ordering \( \preceq_K \) induced by the cone \( K := \mathbb{R}_+^{\mathbb{N} \cup \{\infty\}} \), and \( \bar{x} \in X \) is a weak efficient solution if \( \forall \bar{x} \in A = G^{-1}(-S) : \)

\[
F(x) \notin F(\bar{x}) - \text{int } K, \ \forall x \in A,
\]

i.e.,

\[
A \subset B := \{x \in X : F(x) \notin F(\bar{x}) - \text{int } K\} \quad (3)
\]
Consider the vector infinite optimization problem

\[(VIP) \quad \text{Wmin } \{F(x) : G(x) \in -S\}\]

where:

- $X$, $Y$, $Z$ are lcHtvs.
- $Y$ is equipped with the partial ordering $\preceq_K$ induced by a nonempty, closed, pointed, convex cone $K$ such that $\text{int } K \neq \emptyset$; this ordering is extended to $Y$.
- Similarly, $Z$ is another lcHtvs equipped with the partial ordering $\preceq_S$ induced by a convex cone $S$ such that $\text{int } S = \emptyset$, which is extended to $Z$.
- $F : X \to Y$ and $G : X \to Z$. 

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Consider the vector infinite optimization problem

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where:

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Vector infinite Farkas-type lemmas

- Consider the *vector infinite optimization* problem

$$(\text{VIP}) \quad \text{Wmin} \{ F(x) : G(x) \in -S \}$$

where:
- $X, Y, Z$ are lCHtvs.
- $Y$ is equipped with the partial ordering $\leq_K$ induced by a nonempty, closed, pointed, convex cone $K$ such that $\text{int} K \neq \emptyset$; this ordering is extended to $Y^* := Y \cup \{\pm \infty_Y\}$. 

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Consider the vector infinite optimization problem

\[(\text{VIP}) \quad \text{Wmin} \{ F(x) : G(x) \in -S \}\]

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- $Y$ is equipped with the partial ordering $\leq_K$ induced by a nonempty, closed, pointed, convex cone $K$ such that $\text{int} \ K \neq \emptyset$; this ordering is extended to $Y^\bullet := Y \cup \{\pm \infty_Y\}$.
- Similarly, $Z$ is another lcHtvs equipped with the partial ordering $\leq_S$ induced by a convex cone $S \neq \emptyset$, which is extended to $Z^\bullet := Z \cup \{\pm \infty_Z\}$. 

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Consider the vector infinite optimization problem

\[(VIP) \quad \text{Wmin } \{F(x) : G(x) \in -S\}\]

where:

- \(X, Y, Z\) are lCltvs.
- \(Y\) is equipped with the partial ordering \(\leq_K\) induced by a nonempty, closed, pointed, convex cone \(K\) such that \(\text{int } K \neq \emptyset\); this ordering is extended to \(Y^\bullet := Y \cup \{\pm \infty_Y\}\).
- Similarly, \(Z\) is another lCltvs equipped with the partial ordering \(\leq_S\) induced by a convex cone \(S \neq \emptyset\), which is extended to \(Z^\bullet := Z \cup \{\pm \infty_Z\}\).
- \(F : X \to Y^\bullet\) and \(G : X \to Z^\bullet\).
Different vector Farkas-type results have been given in Dinh-G-López-Mo (2019) where the key tool is the conjugate calculus with vector functions (Tanino, 1992). In the next result, $0_L$ is the null element of the space $L(X, Y)$ of linear continuous mappings from $X$ to $Y$ while the other symbols represent extensions of homonymous concepts from the scalar to the vector setting.

**Theorem (An abstract vector Farkas lemma)**

Let $\bar{x} \in A \cap \text{dom } F$. TFA:
(i) $G^{-1}(-S) \subseteq \{x \in X : F(x) \notin F(\bar{x}) - \text{int } K\}$.
(ii) $0_L \in \partial(F + I_A)(\bar{x})$.
(iii) $(0_L, -F(\bar{x})) \in \text{epi}_K (F + I_A)^*$.

Recall that (i) is nothing else than optimality of $\bar{x}$. 

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Finally, the characterization of $\text{epi}_K (F + I_A)$ in Dinh-G-Long-López (2017) provides Farkas lemmas involving $F$ and $G$ (the data) instead of $F$ and $A$, paying for it the price of aggregating some assumptions.
Finally, the characterization of $\text{epi}_K (F + I_A)$ in Dinh-G-Long-López (2017) provides Farkas lemmas involving $F$ and $G$ (the data) instead of $F$ and $A$, paying for it the price of aggregating some assumptions.

In fact, under suitable assumptions, one has

$$\text{epi}_K (F + I_A)^* = \text{cl} \left( \bigcup_{T \in \mathcal{L}_+(S,K)} \text{epi}_K (F + T \circ G)^* \right),$$

or even

$$\text{epi}_K (F + I_A)^* = \bigcup_{T \in \mathcal{L}_+(S,K)} \text{epi}_K (F + T \circ G)^*,$$

where

$$\mathcal{L}_+(S,K) := \{ T \in \mathcal{L}(Z,Y) : T(S) \subset K \}$$

denotes the \textit{cone of positive operators}. 

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Thank you for your attention!