Multiple recurrence for polynomial and non-polynomial sequences

Joel Moreira
joel.moreira@warwick.ac.uk

University of Warwick

17 July 2020

Based on joint work with

Vitaly Bergelson
Ohio State University

Florian Richter
Northwestern University

Theorem (Furstenberg, 1977)

For any m.p.s. (X, μ, T), *any* $A \subset X$ *with* $\mu(A) > 0$ *and any* $k \in \mathbb{N}$ *there exist* $n \in \mathbb{N}$ *such that*

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-kn}A) > 0.$$

H. Furstenberg, Journal d'Analyse Mathématique, 1977

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Corollary (Szemerédi's theorem)

If $E \subset \mathbb{N}$ *has* $\bar{d}(E) > 0$ *then for every* $k \in \mathbb{N}$ *there exist* $a, n \in \mathbb{N}$ *such that* $\{a, a+n, a+2n, \ldots, a+kn\} \subset E$.

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Theorem (Host-Kra, 2005)

For any m.p.s. (X, μ, T)*, any* $f_1, \ldots, f_k \in L^\infty(X)$*,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \cdot \ldots \cdot T^{kn} f_k \text{ exists in } L^2.$$

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Theorem (Bergelson-Leibman)

Let $p_1, \ldots, p_k \in \mathbb{Z}[x]$ have $p_i(0) = 0$. Then for every m.p.s. (X, μ, T) and every $A \subset X$ with $\mu(A) > 0$ there exist $n \in \mathbb{N}$ s.t.

$$\mu\big(A \cap T^{p_1(n)}A \cap \cdots \cap T^{p_k(n)}A\big) > 0.$$

Bergelson and Leibman, J. Amer. Math. Soc., 1996

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Definition

We say that $p_1, \ldots, p_k \in \mathbb{Z}[n]$ are *jointly intersective* if for every $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ with all of $p_1(n), \ldots, p_k(n)$ divisible by m.

Theorem (Bergelson-Leibman-Lesigne)

Let $p_1, \ldots, p_k \in \mathbb{Z}[x]$. *Then for every m.p.s.* (X, μ, T) *and every* $A \subset X$ *with* $\mu(A) > 0$ *there exist* $n \in \mathbb{N}$ *s.t.*

$$\mu\big(A \cap T^{p_1(n)}A \cap \cdots \cap T^{p_k(n)}A\big) > 0,$$

if and only if $p_1, \ldots, p_k$ *are jointly intersective.*

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Definition

- Two functions $f, g : \mathbb{R} \to \mathbb{R}$ are equivalent if $f(x) = g(x)$ for all large enough x.
- The set B of all equivalent classes forms a ring under pointwise addition and multiplication.
- A *Hardy field* is any subfield $\mathcal{H}$ of B which is closed under taking derivatives.
- If $f : \mathbb{R} \to \mathbb{R}$ is a (representative of a) member of a Hardy field H, we say that f is a *Hardy field function*.

Example

Let $a, b \in \mathbb{R}$.

$$f(x) \in \mathbb{R}[x] \qquad f(x) = x^a \qquad f(x) = x^a \log^b(x)$$

$$f(x) = x^a \zeta(x) \qquad f(x) = e^{\sqrt{\log x}}\Gamma(x) \qquad f(x) = e^{\sqrt{x}}.$$

Write $f \prec g$ if $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 0$.

Theorem (Frantzikinakis)

Let $f_1 \prec \ldots \prec f_k \in \mathcal{H}$. Suppose $\forall i\ \exists \ell \in \mathbb{N}$ such that

$$t^{\ell-1}\log(t) \prec f_i(t) \prec t^{\ell}.$$

Then for any m.p.s. (X, μ, T) and any $A \subset X$ with $\mu(A) > 0$ there exist $n \in \mathbb{N}$ such that

$$\mu\big(A \cap T^{\lfloor f_1(n)\rfloor}A \cap \cdots \cap T^{\lfloor f_k(n)\rfloor}A\big) > 0.$$

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$$\mathrm{span}_{\mathbb{R}}(f_1, \ldots, f_k) = \big\{c_1 f_1 + \cdots + c_k f_k : (c_1, \ldots, c_k) \in \mathbb{R}^k\big\}$$

Conjecture

The same conclusion holds if we only assume that $f_1, \ldots, f_k \in \mathcal{H}$ have polynomial growth and for any nonzero $f \in \mathrm{span}_{\mathbb{R}}(f_1, \ldots, f_k)$ and $p \in \mathbb{Z}[n]$, $\lim_{t\to\infty} |f(t) - p(t)| = \infty$.

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Let $\mathrm{poly}(f_1, \ldots, f_k) :=$

$$\Big\{ p \in \mathbb{R}[t] : \exists f \in \mathrm{span}(f_1, \ldots, f_k) \text{ with } \lim_{t \to \infty} |f(t) - p(t)| = 0 \Big\}.$$

Theorem (Bergelson-M.-Richter)

Let $f_1, \ldots, f_k$ be functions of polynomial growth from a Hardy field and assume that at least one of the following two conditions holds:

1. *For all $q \in \mathbb{Z}[t]$ and non-zero $f \in \mathrm{span}(f_1, \ldots, f_k)$ we have $\lim_{t \to \infty} |f(t) - q(t)| = \infty$.*
2. *There is a jointly intersective collection of polynomials $q_1, \ldots, q_\ell \in \mathbb{Z}[t]$ such that $\mathrm{poly}(f_1, \ldots, f_k) \subset \mathrm{span}(q_1, \ldots, q_\ell)$.*

Then for any m.p.s. (X, μ, T) and any $A \subset X$ with $\mu(A) > 0$ there exist $n \in \mathbb{N}$ such that $\mu\big(A \cap T^{[f_1(n)]} A \cap \cdots \cap T^{[f_k(n)]} A\big) > 0$.

Here $[x]$ is the closest integer to x, i.e., $[x] = \lfloor x + 1/2 \rfloor$.

Corollary

Let $f_1, \ldots, f_k$ be Hardy field functions of polynomial growth and assume that every $p \in \mathrm{poly}(f_1, \ldots, f_k)$ satisfies $p(0) = 0$.
Then for any m.p.s. (X, μ, T) and any $A \subset X$ with $\mu(A) > 0$ there exist $n \in \mathbb{N}$ such that $\mu\big(A \cap T^{[f_1(n)]}A \cap \cdots \cap T^{[f_k(n)]}A\big) > 0$.

Corollary

Let $f_1, \ldots, f_k$ *be Hardy field functions of polynomial growth and assume that every* $p \in \mathrm{poly}(f_1, \ldots, f_k)$ *satisfies* $p(0) = 0$.
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Example

- Let $f_i(t) = t^{c_i}$ where c_i is an arbitrary positive real number.
- Let $f_i(y) = \sum_{j=0}^{d} a_{i,j} t^{c_{i,j}}$, $a_{i,j} \in \mathbb{R}$, $c_{i,j} \in \mathbb{R}^{>0}$.
- Let $\log_1(t) = \log(t)$ and $\log_m(t) = \log(\log_{m-1}(t))$. Let K be the smallest algebra of functions containing t^c for any $c > 1$ and $\log_m^r$ for every $m \in \mathbb{N}$ and $r > 0$. Let $f_1, \ldots, f_k \in K$.

Theorem (Bergelson-M-Richter, 2017)

Let $f \in \mathcal{H}$ satisfy $t^\ell \prec f(t) \prec t^{\ell+1}$ and let $m \in \mathbb{N}$. Then for any m.p.s. (X, μ, T) and any $A \subset X$ with $\mu(A) > 0$ $\exists$ $n \in \mathbb{N}$ s.t.

$$\mu\big(A \cap T^{\lfloor f(n) \rfloor} A \cap T^{\lfloor f(n+1) \rfloor} A \cap \cdots \cap T^{-\lfloor f(n+m) \rfloor} A\big) > 0.$$

The set of such n contains arbitrarily long intervals.

Bergelson, Moreira and Richter, Advances in Mathematics, 2020

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Corollary (of the new theorem)

Let $f_1, \ldots, f_k \in \mathcal{H}$ be of polynomial growth satisfying some technical "non-polynomiality" condition. Then for any m.p.s. (X, μ, T) and any $A \subset X$ with $\mu(A) > 0$ the set of $n \in \mathbb{N}$ such that $\mu\big(A \cap T^{-\lfloor f_1(n) \rfloor} A \cap \cdots \cap T^{-\lfloor f_k(n) \rfloor} A\big) > 0$ contains arbitrarily long intervals.

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$$\mu\left(A \cap \bigcap_{i=1}^{k} \bigcap_{j=0}^{m} T^{\lfloor f_i(n+j) \rfloor} A\right) > 0.$$

Bergelson, Moreira and Richter, Advances in Mathematics, 2020