Group Characters via Ergodic Actions

Kostya Medynets

United States Naval Academy

Group Characters

- G is Abelian. Homomorphisms from G into $\mathbb{T}$ are *characters*.
- G is non-Abelian. Consider a unitary representation $\pi : G \to U(H)$, $dim(H) < \infty$. The function

$$f(g) = \frac{1}{dim(H)} Trace(\pi(g))$$

  is called a *character* of G.
- Extremal characters correspond to irreducible representations.
- Irreducible representations of Abelian groups are 1-dimensional. Characters are homomorphisms into $\mathbb{T}$.

Group Characters "similar" to $f(g) = \frac{1}{dim(H)} Trace(\pi(g))$

1. $f(1) = 1$
2. $f(ab) = f(ba)$ or $f(a) = f(bab^{-1})$ for all $a, b \in G$.
3. The function f is *positive-definite*, that is, for any $\{g_1, \dots, g_n\} \subset G$ and any complex numbers $\{c_1, \dots, c_n\}$,

$$\sum_{i,j} \bar{c}_i c_j f(g_i^{-1} g_j) \geq 0.$$

- Take a unitary representation $\pi : G \to \mathrm{U}(H)$. Then $f(g) =< \pi(g)\xi, \xi >$ is positive semi-definite.
- Conversely, the matrix $f(g_i^{-1} g_j)$ defines an inner product in the space $Fun(G, \mathbb{C})$).

If G is *infinite*, then any function $f : G \to \mathbb{C}$ satisfying (1), (2), (3) is called a *character*.

M.P. Actions and Characters

- $G \curvearrowright (X, \mu)$
- Vershik: $f(g) = \mu(Fix(g))$ **is a character,** $Fix(g) = \{x : g(x) = x\})$
- $f(1) = \mu(X) = 1$
- $f(aba^{-1}) = \mu(Fix(aba^{-1})) = \mu(a(Fix(b))) = f(g)$
- Orbit equivalence relation $R = \{(x, y) : x = gy \text{ for some } g \in G\} \subset X \times X$.
- For $A \subset X \times X$, define $\bar{\mu}(A) = \int_X card(A_x) d\mu(x)$
- Define $\pi : G \to U(L^2(X \times X, \bar{\mu}))$ by
$$(\pi(g)f)(x, y) = f(g^{-1}x, y).$$
- $$(\pi(g)1_\Delta, 1_\Delta) = \int_{X \times X} 1_\Delta(g^{-1}x, y) 1_\Delta(x, y) d\bar{\mu} = \mu(x : g(x) = x)$$

M.P. Actions and Characters

Every group always admits at least two characters

- the regular character $\delta(g) = 0$ if $g \neq 1$ and $\delta(1) = 0$. For ICC groups, the regular character is extremal.
- the identity character $\rho(g) = 1$ for every g.

Theorem (Dudko-Medynets)
If a countable group G has no non-trivial characters, then any ergodic action of G on a probability measure space is essentially free, i.e. $\mu(Fix(g)) = 0$ for every $g \neq e$.

Example: The Infinite Special Linear Group

- Consider a finite field $\mathbb{F}_q$ and the set of matrices

$$M(\infty) = \{I_\infty + A : A \in \mathrm{Mat}_n(\mathbb{F}_q)) \text{ for some } n\}$$

- $SL(\infty) = \{A : det(A) = 1\}$.
- $SL(\infty)$ acts on the linear space $X = \mathbb{F}_q^{\mathbb{N}}$
- (X, G) has a unique fixed point $(0, 0, \ldots,)$. The orbit equivalence relation is almost the cofinal relation.
- Given $g \in G$,

$$\mu(\{x : gx = x\}) = \mu(\{x : (g - I)x = 0\}) = \left(\frac{1}{q}\right)^{rnk(g-I)}.$$

Theorem (Skudlarek 1976, Dudko-M)
Let χ be a non-trivial indecomposable character of G, then it comes from the diagonal action of G on $(X^k, \mu^{\times k})$ for some k, i.e. $\chi(g) = \left(\frac{1}{q}\right)^{k \times rnk(g-I)}$.

Bratteli Diagram

A **Bratteli diagram** is a graded graph (V, E) whose vertices and edges are partitioned into $V = \bigcup_{n\geq 0} V_n$ and $E = \bigcup_{n\geq 1} E_n$ and the edges E_n connect the vertices of V_{n-1} to V_n.

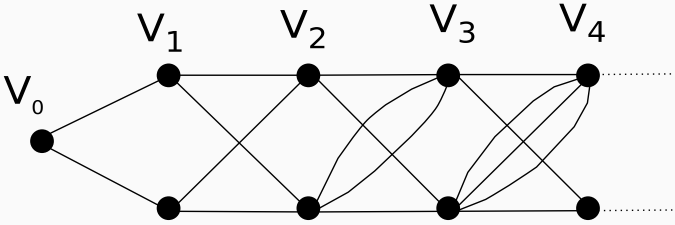
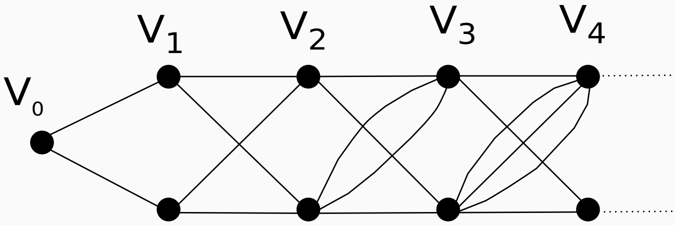


Figure 1: Diagrammatic representation of a Bratteli diagram.

Full Groups of Bratteli Diagrams



- X is the set of infinite paths in the diagram.
- For $n \geq 1$ and $v \in V_n$, denote by $G_v^{(n)}$ the group of transformations of X that permute the initial segments of the paths going through the vertex v. Set

$$G^{(n)} = \prod_{v \in V_n} G_v^{(n)} = \prod_{v \in V_n} Sym(\text{ number of paths between } V_0 \text{ and } v).$$

Characters of Full Groups

- Consider a Bratteli diagram B. Let G be the associated full group and X be the path-space.
- Suppose that
    - the group G is simple;
    - the dynamical system (X, G) admits only finitely many ergodic measures $\mu_1, \ldots, \mu_k$.
- For each k-tuple $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}^k_{\geq 0}$, set

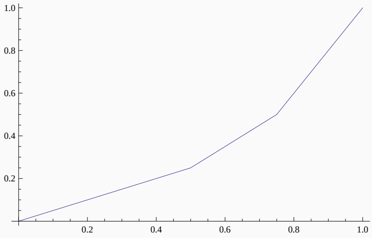
$$X^{|\alpha|} = X^{\alpha_1} \times \cdots \times X^{\alpha_k} \text{ and } \mu^\alpha = \mu_1^{\alpha_1} \times \cdots \times \mu_k^{\alpha_k}.$$

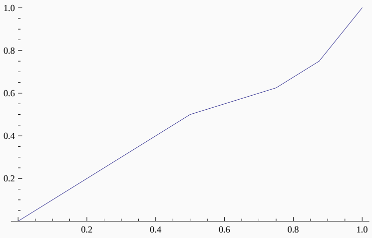
- Consider the diagonal action of G on $(X^{|\alpha|}, \mu^\alpha)$.

Theorem (Dudko-Medynets)
Every extremal character $f : G \to \mathbb{C}$ is either regular or of the form $f(g) = \mu^\alpha(Fix(g))$ for some diagonal action of G on $(X^{|\alpha|}, \mu^\alpha)$.

Additional Examples: Thompson's group

Thompson's group F is generated by the functions





The group F consists of all piecewise linear homeomorphisms from the closed interval $[0, 1]$ to itself that are differentiable except at finitely many dyadic rational numbers and such that on the intervals of differentiability the derivatives are powers of 2. The elements of F preserve orientation.

The Commutator Subgroup F'

Denote by F' the commutator subgroup of F. The group F' consists of all homeomorphisms from F that are trivial in neighborhoods of 0 and 1.

- F' is simple. For simple groups, if f is an extreme character, then $|f(g)| < 1$ whenever $g \neq 1$.
- The action of F' on $X = (0, 1)$ has no probability invariant measure.

Theorem
(Dudko-Medynets) The group F' has no non-trivial characters.