Predictive Sets

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Predictive sets

A set $P \subset \mathbb{N}$ is called a predictive set if for all zero entropy processes $X_j; j \in \mathbb{Z}$, X_0 is measurable with respect to $X_i; i \in P$.

Alternatively, a set is called a predictive set if

$$H(X_0 \mid X_i; i \in P) = 0$$

for all zero entropy processes $X_j; j \in \mathbb{Z}$ with a finite state space.

Notation: X_P

Given $P \subset \mathbb{Z}$ we will write X_P to mean $X_i; i \in \mathbb{P}$.

So $X_{\mathbb{Z}}$ will denote the entire process.

The first example: $P = \mathbb{N}$ is predictive

We shall assume here on that all processes $X_{\mathbb{Z}}$ are stationary and have finite state space.

Let $h(X_{\mathbb{Z}})$ denote the Kolmogorov-Sinai entropy of the process $X_{\mathbb{Z}}$.

The Kolmogorov-Sinai entropy can be computed using the conditional entropy formula:

$$h(X_{\mathbb{Z}}) := H(X_0 \mid X_{\mathbb{N}}).$$

Thus $X_{\mathbb{Z}}$ has zero entropy if and only if $H(X_0 \mid X_{\mathbb{N}}) = 0$. $\mathbb{N}$ is a predictive set.

The next realisation: $P = k\mathbb{N}$ is predictive

We know that $X_{\mathbb{Z}}$ has zero entropy if and only if $X_{k\mathbb{Z}}$ has zero entropy.

Thus for any zero entropy process $X_{\mathbb{Z}}$, we have that $H(X_0 \mid X_{k\mathbb{N}}) = 0$ for all k.

As a consequence we have that $k\mathbb{N}$ is predictive for all $k \in \mathbb{N}$.

But $P = k\mathbb{N} + r$ is not predictive (when $r \not\equiv 0 \pmod{k}$)

Let us see why this is true for $k = 2$ and $r = 1$.

Take two independent random variables Y_1 taking values -1 or 1 and Y_2 taking values 2 or -2 with equal probability.

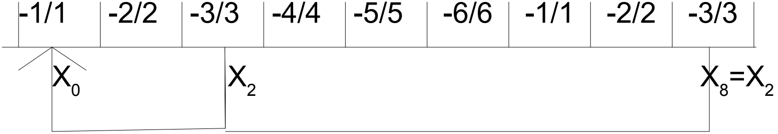
Now consider a process $X_{\mathbb{Z}}$ for which (independent of Y_1 and Y_2)

$$\text{with probability } 1/2,\ X_{2\mathbb{N}} := Y_1;\ X_{2\mathbb{N}+1} := Y_2 \quad \text{and}$$
$$\text{with probability } 1/2,\ X_{2\mathbb{N}} := Y_2;\ X_{2\mathbb{N}+1} := Y_1.$$

Clearly $X_{\mathbb{Z}}$ has zero entropy but

$$\mathbb{P}(X_0 > 0 \mid X_{2\mathbb{N}+1}) = 1/2.$$

$P = k\mathbb{N} + r$ is not predictive (when $r \not\equiv 0 \pmod{k}$)



We cannot predict whether X_0 will be -1 or 1 even if know what X_2, X_8 ... are.

We can also construct a zero entropy weak mixing process $X_{\mathbb{Z}}$ for which

$$H(X_0 \mid X_{k\mathbb{N}+r}) \neq 0.$$

Return-time sets are predictive

Let (X, μ, T) be a probability preserving transformation (ppt). Given a set $U \subset X$ of positive measure, we denote by

$$N(U, U) := \{n \in \mathbb{N} \ : \ \mu(T^n(U) \cap U) > 0\}$$

the return-time set U.

A set $A \subset \mathbb{N}$ is called a return-time set if $A = N(U, U)$ for some ppt.

Theorem (Chandgotia, Weiss)

Return-time sets are predictive sets.

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Return-time sets are predictive sets.

1. $k\mathbb{N}$ is the return-time set for the transformation $T : \mathbb{Z}/k\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$ given by $T(i) = i + 1$.
2. If (X, μ, T) is a zero entropy ppt and $U \subset X$ is such that $\mu(U) > 0$ then for all predictive sets P, $P \cap N(U, U)$ is also predictive.
3. In particular, for predictive sets P, $\alpha \in \mathbb{R}/\mathbb{Z}$ and $\epsilon > 0$

$$P \cap \{n \ : \ n\alpha \in (-\epsilon, \epsilon)\}$$

is predictive.

$SIP^\star$ sets

Given a sequence of natural numbers $S = \{s_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$, we write

$$SIP(S) := \left\{ \sum_{i=1}^{\infty} \epsilon_i s_i \ : \ \epsilon_i \in \{-1, 0, 1\} \right\} \cap \mathbb{N}.$$

A set $P \subset \mathbb{N}$ is called $SIP^\star$ if it intersects every SIP set.

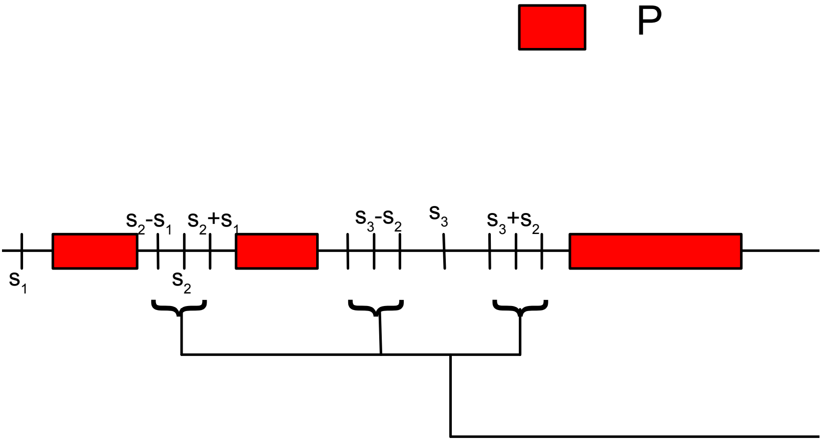
$SIP^\star$ sets

$$SIP(S) := \left\{ \sum_{i=1}^{\infty} \epsilon_i s_i \;:\; \epsilon_i \in \{-1, 0, 1\} \right\} \cap \mathbb{N}.$$

1. $k\mathbb{N}$ is $SIP^\star$: Given a sequence $S = \{s_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$ there exists a subsequence $s_{i_1}, s_{i_2}, \ldots, s_{i_k}$ (which are equal modulo k) such that
$$\sum_{t=1}^{k} s_{i_t} \in k\mathbb{N}.$$
Thus $SIP(S) \cap k\mathbb{N} \neq \emptyset$.
2. if $r \not\equiv 0 \pmod{k}$ then $k\mathbb{N} + r$ is not SIP^*: If a sequence $S \subset k\mathbb{N}$ then $SIP(S) \subset k\mathbb{N}$.
3. $SIP^\star$ sets have bounded gaps.

$SIP^\star$ sets have bounded gaps.

Suppose P is a set such that it does not have bounded gaps. Then we can fit an SIP set in its complement.



Predictive sets are $SIP^\star$

Theorem

Predictive sets are $SIP^\star$.

Thus predictive sets have bounded gaps.

In fact, for all $SIP(S)$ there exists a weak mixing zero entropy process $X_\mathbb{Z}$ such that

$$X_0 \text{ is independent of } X_i \text{ for } i \in \mathbb{N} \setminus SIP(S).$$

A few questions

If P is a predictive set, $\epsilon > 0$ and $\alpha \in \mathbb{R}/\mathbb{Z}$ then

$$\{n \in \mathbb{N} \ : \ n\alpha \in (-\epsilon, \epsilon)\} \cap P$$

is predictive.

Question

Is the intersection of two predictive sets also predictive? Is the intersection non-empty?

We proved that predictive sets are SIP^*.

Question

Are all $SIP^\star$ sets predictive?

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Is the intersection of two predictive sets also predictive? Is the intersection non-empty?

Question

Are all SIP* sets predictive?

Question

Take an irrational number $\alpha \in \mathbb{R}/\mathbb{Z}$. Is the set

$$\{n \in \mathbb{N} \ : \ n\alpha \in (0, \epsilon)\}$$

predictive for some $\epsilon < 1/2$?

Question

Take an irrational number $\alpha \in \mathbb{R}/\mathbb{Z}$*. Is the set*

$$\{n \in \mathbb{N} \ : \ n\alpha \in (0, \epsilon)\}$$

predictive for some $\epsilon < 1/2$*?*

If the answer is yes then we have two predictive sets

$$\{n \in \mathbb{N} \ : \ n\alpha \in (0, \epsilon)\} \text{ and } \{n \in \mathbb{N} \ : \ -n\alpha \in (0, \epsilon)\}$$

which do not intersect.

If the answer is no then we have a $SIP^\star$ set which is not predictive. This would follow from the theorem:

Theorem (Akin and Glasner, 2016)

The sets $\{n \in \mathbb{N} \ : \ n\alpha \in (0, \epsilon)\}$ *are* $SIP^\star$*.*

There are predictive sets which do not contain return-time sets.

Consider the set

$$Q = \{n^2 \ : \ n \in \mathbb{N}\}.$$

For all $i, k \in \mathbb{N}$ we have that if

$$n^2 = -i + 3i^2k = i(-1 + 3ik)$$

then since i and $-1 + 3ik$ are prime to each other, they are perfect squares.

But this is impossible because $-1 + 3ik \equiv -1 \pmod 3$. Thus $\mathbb{N} \setminus Q$ contains $-i + 3i^2k; k \in \mathbb{N}$.

There are predictive sets which do not contain return-time sets.

Hence we have that

$$H(X_{-i} \mid X_{\mathbb{N}\setminus Q}) = 0$$

for all $i \in \mathbb{N}$.

But then

$$H(X_i \mid X_{\mathbb{N}\setminus Q}) = 0$$

for all $i \in \mathbb{Z}$.

It is well known that any return-time set must intersect the set $\{n^2 \ : \ n \in \mathbb{N}\}$. Thus there are predictive sets which are not return-time sets.

Predictive sets

Question

Let $\{n_k\}_{k \in \mathbb{N}}$ *be an increasing sequence such that* $n_{k+1} - n_k$ *is also an increasing sequence. Prove that*

$$H(X_0 \mid X_{\mathbb{N} \setminus \{n_k \mid k \in \mathbb{N}\}}) = 0.$$

We do not know this even in the case $n_k = k^3$. The only partial result uses Fermat's last theorem.

Connections with harmonic analysis

Such a result would have many interesting consequences. For instance:

Theorem

If $P \subset \mathbb{N}$ is a set such that $P + i$ is predictive for all $i \in \mathbb{N}$ then for all singular measures μ on $\mathbb{R}/\mathbb{Z}$ there exists $p \in P$ such that the Fourier coefficient

$$\hat{\mu}(p) \neq 0.$$

In other words any measure μ on $\mathbb{R}/Z$ whose Fourier coefficients are supported on $\mathbb{Z} \setminus P$ must have an absolutely continuous component.

This is very close to Riesz sets as defined by Yves Myer: A set $Q \subset \mathbb{Z}$ is called a Riesz set if all measures on $\mathbb{R}/\mathbb{Z}$ whose Fourier coefficients are supported on Q are absolutely continuous.

Summary

Return-time sets are predictive.

The converse is not true.

Predictive sets are $SIP^\star$.

Predictive sets have bounded gaps.

Questions

1. Is the intersection of two predictive sets also a predictive set?
2. Are all $SIP^\star$ sets predictive?
3. Is $\{n \,:\, n\alpha \in (0,\epsilon)\}$ a predictive set for irrational α?
4. Let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence such that $n_{k+1} - n_k$ is also an increasing sequence. Prove that

$$H(X_0 \mid X_{\mathbb{N}\setminus\{n_k \mid k\in\mathbb{N}\}}) = 0.$$