Topological characteristic factors and nilsystems

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July 17, 2020

Outline

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Interplay between measurable and topological dynamics

"The two sister branches of the theory of dynamical systems called **ergodic theory** (or measurable dynamics) and **topological dynamics** use these words to describe different but parallel notions in their respective theories and the surprising fact is that many of the corresponding results are rather similar." [1]

——- Glasner-Weiss

[1] E. Glasner and B. Weiss, *On the interplay between measurable and topological dynamics*, Handbook of dynamical systems. Vol.1B, 597-648, 2006.

Recurrence theorems

Theorem (Poincaré Recurrence Theorem)

Let $(X, \mathscr{X}, \mu, T)$ *be a m.p.s. Let* $A \in \mathscr{X}$ *with* $\mu(A) > 0$*. Then there exits* $n \in \mathbb{N}$ *such that*

$$\mu(A \cap T^{-n}A) > 0.$$

Theorem (Birkhoff Recurrence Theorem)

Let (X, T) *be a t.d.s. Then there exits some* $x \in X$ *and a sequence* $\{n_i\} \subseteq \mathbb{N}$ *such that*

$$T^{n_i}x \to x, \quad i \to \infty.$$

Multiple Recurrence theorems

Theorem (Multiple Recurrence Theorem, Furstenberg-Weiss, 1978)

Let (X,T) be a t.d.s. and $d \in \mathbb{N}$. Then there exits some $x \in X$ and a sequence $\{n_i\} \subseteq \mathbb{N}$ such that

$$T^{n_i}x \to x, T^{2n_i}x \to x, \ldots, T^{dn_i}x \to x, \quad i \to \infty.$$

• Equivalently, let (X,T) be a minimal system and $d \in \mathbb{N}$. Then for every non-empty open subset $U \subset X$, there exists n such that

$$U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset.$$

Theorem (Van der Waerden theorem, 1927)

For any $r \in \mathbb{N}$ and any partition $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, one of C_i contains arbitrarily long arithmetic progressions.

Multiple Recurrence theorems

Theorem (Multiple Recurrence Theorem, Furstenberg, 1977)

Let $(X, \mathscr{X}, \mu, T)$ be a m.p.s. and $d \in \mathbb{N}$. Let $A \in \mathscr{X}$ with $\mu(A) > 0$. Then there exits $n \in \mathbb{N}$ such that

$$\mu(A \cap T^{-n}A \cap \ldots \cap T^{dn}A) > 0.$$

Theorem (Szemerédi theorem, 1975)

Any subset of $\mathbb{N}$ with positive upper density contains arbitrarily long arithmetic progressions.

Multiple ergodic average

Theorem (Furstenberg, 1977)

Given a m.p.s. $(X, \mathscr{X}, \mu, T)$, $d \in \mathbb{N}$ *and* $A \in \mathscr{X}$ *with* $\mu(A) > 0$, *one has*

$$\liminf_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^{-n}A \cap \cdots \cap T^{-dn}A) > 0.$$

In particular, there exists some $n \in \mathbb{N}$ *such that*

$$\mu(A \cap T^{-n}A \cap \cdots \cap T^{-dn}A) > 0.$$

Furstenberg correspondence principle tells us:

Szemerédi Theorem $\Longleftrightarrow \{n : \mu(A \cap T^{-n}A \cap \cdots \cap T^{-dn}A) > 0\} \neq \emptyset.$

Multiple ergodic average

Multiple ergodic average

Let $(X, \mathscr{X}, \mu, T)$ be a measure-preserving system, $d \in \mathbb{N}$ and $f_1, f_2, \cdots, f_d \in L^\infty(X, \mu)$. Is

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) f_2(T^{2n} x) \cdots f_d(T^{dn} x)$$

- convergent in $L^2(X, \mu)$?
- convergent for μ-a.e. $x \in X$?

When $d = 1$:

- von Neumann mean ergodic theorem (1932);
- Birkhoff pointwise ergodic theorem (1931)

Characteristic factor

Definition (Furstenberg-Weiss, 1977, 1996)

Let $(X,\mathscr{X},\mu,T)$ be a m.p.s. and $(Y,\mathscr{Y},\nu,T)$ be a factor of X. For $d\geq 1$, we say that Y is a **characteristic factor** of X if for all $f_1,\ldots,f_d\in L^\infty(X,\mu)$,

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^n x)\ldots f_d(T^{dn}x)-\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}(f_1|\mathscr{Y})(T^n x)\ldots\mathbb{E}(f_d|\mathscr{Y})(T^{dn}x)\to 0$$

in $L^2(X,\mu)$.

- The key point lies in:
  to show L^2-ergodic average theorem of order d for $(X,\mathscr{X},\mu,T)$ $\Longleftrightarrow$ to show L^2-ergodic average theorem of order d for its characteristic factor of order d (Z_d,μ_d,T_d).

Characteristic factor and nilsystem

Theorem (Host-Kra, 2005; Ziegler, 2007)

The characteristic factor of order d $(Z_d, \mathscr{Z}_d, \mu_d, T_d)$ of a m.p.s. $(X, \mathscr{X}, \mu, T)$ is an inverse limit of d-step nilsystem.

Structure Theorem + Theory of nilsystems
$\Rightarrow L^2$ multiple ergodic theorem !

Questions

- What is the topological characteristic factor of a topological system?
- How to induce the problem on a topological system to its nilfactors?

Odd recurrence

Theorem (Host-Kra (d=3), 2002; Frantzikinakis, 2004)

Let $(X, \mathscr{X}, \mu, T)$ be a m.p.s. and $d \in \mathbb{N}$. Assume that T^k is ergodic for some integer $k \geq 2$ and let E be a set with $\mu(E) > 0$. Let $0 \leq j < k$. Then there exists $n \equiv j \pmod{k}$ such that

$$\mu(E \cap T^{-n}E \cap T^{-2n}E \cap \ldots \cap T^{-dn}E) > 0.$$

Theorem (Host-Kra (d=3), 2002; Frantzikinakis, 2004)

Let $(X, \mathscr{X}, \mu, T)$ be a m.p.s. and $d \in \mathbb{N}$. Assume that T^k is ergodic for some integer $k \geq 2$. Let $a_1, \ldots, a_d$ be non-zero distinct integers and let $f_1, \ldots, f_d \in L^\infty(X, \mu)$. Then the limits

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_1(T^{ka_1 n}x) \ldots f_d(T^{ka_d n}x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_1(T^{a_1 n}x) \ldots f_d(T^{a_d n}x)$$

exist in $L^2(X, \mu)$.

Odd recurrence

Question

Let (X, T^k) be minimal for some $k \geq 2$ and $d \in \mathbb{N}$. Let (X, T^k) be minimal for some $k \geq 2$ and $d \in \mathbb{N}$. Is it true that for any non-empty open subset U of X and $0 \leq j < k$ one has

$$U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset,$$

for some $n \equiv j \pmod{k}$?

★ Equivalently, is there a sequence $\{n_i\}$ with $n_i \equiv j \pmod{k}$ such that

$$T^{n_i}x \to x,\ T^{2n_i}x \to x, \ldots,\ T^{dn_i}x \to x,\ i \to \infty$$

for x in a dense G_δ subset of X.

Furstenberg-Sarközy Theorem

- Furstenberg-Sarközy (1977): $\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f(T^{p(n)}x) = P_{\mathscr{H}_{rat}} f$, $\mathscr{H}_{rat} = \overline{\{f : \exists a \in \mathbb{N}\ s.t.\ T^a f = f\}}$.
- Bourgain (1989): convergence a.e.

Theorem

Let $(X, \mathscr{X}, \mu, T)$ be a totally ergodic m.p.s. Then for all integer valued polynomial $p(n)$ and $f \in L^\infty(X, \mathscr{X}, \mu)$ there is some $X_0 \in \mathscr{X}$ with $\mu(X_0) = 1$ such that for all $x \in X_0$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f(T^{p(n)}x) = \int_X f\, d\mu.$$

Density problem

Problem

Let (X,T) be a totally minimal system and integer valued polynomial $p(n)$. Is there a residual set $X_0 \subseteq X$ such that for each $x \in X_0$

$$\overline{\{T^{p(n)}x : n \in \mathbb{N}\}} = X?$$

Outline

Topological characteristic factor

Let $\phi : (H,T) \to (M,T)$ be a factor map of t.d.s. A subset L of H is called **ϕ-saturated** if

$$L = \phi^{-1}(\phi(L)).$$

Definition

Let $\pi : (X,T) \to (Y,T)$ be a factor map and $d \in \mathbb{N}$. (Y,T) is said to be a **d-step topological characteristic factor (along arithmetic progressions)** if there exists a dense G_δ set X_0 of X such that for each $x \in X_0$ the orbit closure

$$L_x = \overline{orb(\underbrace{(x,\ldots,x)}_{d \text{ times}}, T \times T^2 \times \ldots \times T^d)}$$

is $\pi^{(d)} = \underbrace{\pi \times \ldots \times \pi}_{d \text{ times}}$ saturated.

Theorem (Glasner, 1994)

Up to proximal extensions, a characteristic family for $T \times T^2 \times \ldots \times T^d$ is the family of canonical PI flows of class $d-1$. In particular,

- *if (X,T) is a distal minimal system, then its largest $d-1$-step distal factor is its d-step topological characteristic factor;*
- *if (X,T) is a weakly mixing minimal system, then the trivial system is its d-step topological characteristic factor for all $d \in \mathbb{N}$.*

Conjecture (Glasner)

If (X,T) is a distal minimal system, then its maximal $(d-1)$-step pro-nilfactor is its d-step topological characteristic factor along arithmetic progressions.

Topological characteristic factor

Theorem (Glasner-Huang-S.-Weiss-Ye,2020)

Let (X,T) be a minimal system and X_∞ be its ∞-step nilfactor. If $\pi_\infty : (X,T) \to (X_\infty, T)$ is open, then for all $d \in \mathbb{N}$, $(d-1)$-step pro-nilfactor X_{d-1} is its d-step topological characteristic factor along arithmetic progressions.

Corollary

If (X,T) is a distal minimal system, then its maximal $(d-1)$-step pro-nilfactor is its d-step topological characteristic factor along arithmetic progressions.

Topological characteristic factor

In general, up to almost 1-to-1 extensions, the maximal $(d-1)$-step pro-nilfactor is its d-step topological characteristic factor.

Theorem (Glasner-Huang-S.-Weiss-Ye,2020)

Let (X,T) be a minimal system, and $\pi : X \to X_\infty$ be the factor map. Then there are minimal systems X^* and X_∞^* which are almost one to one extensions of X and X_∞ respectively, and a commuting diagram below such that X_∞^* is a d-step topological characteristic factor of X^* for all $d \geq 2$,

$$\begin{array}{ccc} X & \xleftarrow{\sigma^*} & X^* \\ \Big\downarrow \pi & & \Big\downarrow \pi^* \\ X_\infty & \xleftarrow{\tau^*} & X_\infty^* \end{array}$$

Applications

Theorem (Glasner-Huang-S.-Weiss-Ye,2020)

Let (X,T^k) be minimal for some $k \geq 2$ and $d \in \mathbb{N}$. Then for any $0 \leq j < k$ there is a sequence $\{n_i\}$ with $n_i \equiv j \ (mod\ k)$ such that

$$T^{n_i}x \to x,\ T^{2n_i}x \to x, \ldots,\ T^{dn_i}x \to x,\ i \to \infty$$

for x in a dense G_δ subset of X.

★ It is equivalent to show that for any non-empty open subset U of X and $0 \leq j < k$ one has

$$U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset,$$

for some $n \equiv j \pmod{k}$.

Applications

Theorem (Glasner-Huang-S.-Weiss-Ye,2020)

Let (X,T) be a totally minimal system, and $P(n) = an^2 + bn + c$ be an integral polynomial with $a \neq 0$. Then there is a dense G_δ subset Ω of X such that for every $x \in \Omega$,

$$\overline{\{T^{P(n)}(x) : n \in \mathbb{Z}\}} = X.$$

Weakly mixing case

Theorem (Huang-S.-Ye, 2019)

Let (X,T) be a weakly mixing minimal system and $p_1,\dots,p_d$ be distinct polynomials, taking integer values at integers and $p_i(0)=0, i=1,\dots,d$. Then there is a dense G_δ-set X_0 of X such that for any $x\in X_0$

$$\overline{\{(T^{p_1(n)}(x),\dots,T^{p_d(n)}(x)) : n\in\mathbb{Z}\}} = X^d.$$

Corollary

Let (X,T) be a topologically weakly mixing minimal system and integer valued polynomial $p(n)$. Then there is a residual set $X_0\subseteq X$ such that for each $x\in X_0$

$$\overline{\{T^{p(n)}x : n\in\mathbb{Z}\}} = X.$$

Outline

Ingredients of the proofs

- Regionally proximal relation of order d, $\mathbf{RP}^{[d]}$
- The connection of $\mathbf{RP}^{[d]}$ with multiple recurrence sets
- The system $(N_d(X), \langle \sigma_d, \tau_d \rangle)$
- A Saturation Theorem

Regionally proximal of order d

Definition (Regionally proximal relation of order d, Host-Kra-Maass)

$(x,y) \in \mathbf{RP}^{[d]}(X)$ if for any $\delta > 0$, there exist $x', y' \in X$ and a vector $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ such that $\rho(x,x') < \delta, \rho(y,y') < \delta$, and

$$\rho(T^{\mathbf{n}\cdot\varepsilon}x', T^{\mathbf{n}\cdot\varepsilon}y') < \delta \text{ for any } \varepsilon \in \{0,1\}^d,\ \varepsilon \neq (0,\ldots,0),$$

where $\mathbf{n}\cdot\varepsilon = \sum_{i=1}^d \varepsilon_i n_i$.

- $d = 1$: $\{\mathbf{n}\cdot\varepsilon\} = \{n_1\}$
- $d = 2$: $\{\mathbf{n}\cdot\varepsilon\} = \{n_1, n_2, n_1 + n_2\}$
- $d = 3$: $\{\mathbf{n}\cdot\varepsilon\} = \{n_1, n_2, , n_3, n_1 + n_2, n_1 + n_3, n_2 + n_3, n_1 + n_2 + n_3\}$

Regionally proximal of order d

Theorem

Let (X,T) be a minimal system and $d \in \mathbb{N}$. Then

1. $\mathbf{RP}^{[d]}(X)$ *is an equivalence relation.*
2. $(X_d = X/\mathbf{RP}^{[d]}, T)$ *is the maximal d-step nilfactor of (X,T).*

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \pi\downarrow & & \downarrow\pi \\ X_d & \xrightarrow{T} & X_d \end{array}$$

- distal minimalsystems: Host-Kra-Maass, 2010
- general case: S.-Ye, 2012

Structure of minimal systems:

$$\{pt\} = X_0 \xleftarrow{\pi_0} X_1 \xleftarrow{\pi_1} \cdots \xleftarrow{\pi_{n-1}} X_n \xleftarrow{\pi_n} X_{n+1} \xleftarrow{\pi_{n+1}} \cdots \longleftarrow X_\theta \xleftarrow{\pi_\theta} X.$$

∞-step nilsystem

It follows that for any minimal system (X,T),

$$\mathbf{RP}^{[\infty]} = \bigcap_{d\geq 1} \mathbf{RP}^{[d]}$$

is a closed invariant equivalence relation.

Definition

A minimal system (X,T) is an **∞-step nilsystem** if $\mathbf{RP}^{[\infty]}$ is trivial.

Theorem (Dong-Donoso-Maass-S.-Ye, 2013)

A minimal system is an ∞-step nilsystem if and only if it is an inverse limit of minimal nilsystems.

The connection of $\mathbf{RP}^{[d]}$ with recurrence sets

Definition

1. $S \subset \mathbb{Z}$ is a set of **d-recurrence** if for every measure preserving system $(X, \mathscr{X}, \mu, T)$ and for every $A \in \mathscr{X}$ with $\mu(A) > 0$, there exists $n \in S$ such that $\mu(A \cap T^{-n}A \cap \ldots \cap T^{-dn}A) > 0$.
2. $S \subset \mathbb{Z}$ is a set of **d-topological recurrence** if for every minimal t.d.s. (X, T) and for every nonempty open subset U of X, there exists $n \in S$ such that $U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset$.
3. (Host-Kra) $S \subset \mathbb{Z}$ is a **Nil_d Bohr_0-set**, if there are a d-step nilsystem (X, T), $x \in X$ and a neighbourhood U of x such that $S \supset N(x, U)$.

Let $\mathscr{F}_{Poi_d}$ (resp. $\mathscr{F}_{Bir_d}$, $\mathscr{F}_d$) be the family consisting of all sets of d-recurrence (resp. d-topological recurrence, Nil_d Bohr_0-set).

Theorem (Huang-S.-Ye, 2016)

$$\mathscr{F}_{Poi_d} \subset \mathscr{F}_{Bir_d} \subset \mathscr{F}_d^*.$$

The connection of $\mathbf{RP}^{[d]}$ with recurrence sets

Theorem (Huang-S.-Ye, 2016)

Let (X,T) be a minimal t.d.s., $d \in \mathbb{N}$ and $x,y \in X$. Then the following statements are equivalent:

1. $(x,y) \in \mathbf{RP}^{[d]}(X,T)$.
2. $N_T(x,U) \in \mathscr{F}_{Poi_d}$ *for each neighborhood* U *of* y.
3. $N_T(x,U) \in \mathscr{F}_{Bir_d}$ *for each neighborhood* U *of* y.
4. $N_T(x,U) \in \mathscr{F}_d^*$, *for each neighborhood* U *of* y.

The system $(N_d(X), \langle \sigma_d, \tau_d \rangle)$

Let (X,T) be a t.d.s., $x \in X$, $A \subseteq X$ and $d \in \mathbb{N}$. Set $x^{(d)} = (x,x,\dots,x) \in X^d$,

$$\Delta_d(X) = \{(x,\dots,x) \in X^d : x \in X\},$$

$$\sigma_d = T \times \ldots \times T \ (d \text{ times}),$$

$$\tau_d = \tau_d(T) = T \times T^2 \times \ldots \times T^d.$$

Let

$$N_d(X,T) = N_d(X) = \overline{\mathscr{O}}(\Delta_d(X), \tau_d).$$

If (X,T) is transitive and $x \in X$ is a transitive point, then $N_d(X) = \overline{\mathscr{O}}(x^{(d)}, \langle \sigma_d, \tau_d \rangle)$.

The system $(N_d(X), \langle \sigma_d, \tau_d \rangle)$

Theorem (Glasner, 1994)

Let (X,T) be a minimal system, $d \in \mathbb{N}$. Then the system $(N_d(X), \langle \sigma_d, \tau_d \rangle)$ is minimal.

Corollary (MBR)

Let (X,T) be minimal and $d \in \mathbb{N}$. Then for any non-empty open subset U of X one has

$$U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset.$$

Results on $N_d(X,T)$

Theorem (Glasner-Huang-S.-Weiss-Ye,2020)

Let (X,T) be a minimal system and $d \in \mathbb{N}$. Then the maximal equicontinuous factor of $(N_d(X,T), \langle \sigma_d, \tau_d \rangle)$ is $(N_d(X_1,T), \langle \sigma_d, \tau_d \rangle)$, where as above X_1 is the maximal equicontinuous factor of (X,T).

Theorem (Glasner-Huang-S.-Weiss-Ye,2020)

Let (X,T) be a minimal system and $k \geq 2$. Then (X,T^k) is minimal if and only if $N_d(X,T) = N_d(X,T^k)$ for each $d \in \mathbb{N}$.

Outline

Problems

Problem

Let (X,T) be a totally minimal system and integer valued polynomial $p(n)$. Is there a residual set $X_0 \subseteq X$ such that for each $x \in X_0$

$$\overline{\{T^{p(n)}x : n \in \mathbb{N}\}} = X?$$

Problem

Let (X,T^k) be minimal for some $k \geq 2$ and $d \in \mathbb{N}$. Then for non-constant integral polynomials $P_m(n)$ with $P_m(0) = 0$, $1 \leq m \leq d$, and any $0 \leq j < k$, there is a sequence $\{n_i\}_{i\in\mathbb{N}}$ such that

$$T^{P_1(n_i)}x \to x, \ldots, T^{P_d(n_i)}x \to x, \ i \to \infty,$$

where $n_i \equiv j \pmod{k}$ and x is in a dense G_δ set of X.

Problems

In ergodic theory due to Vinogradov: under the total ergodicity assumption, for all $f \in L^2(X,\mu)$,

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p\leq N, p \text{ prime}} T^p f = \int_X f d\mu \quad \text{in } L^2(X,\mu),$$

where $\pi(N)$ denotes the number of primes less than or equal to N.

Problem

Let (X,T) be a totally minimal system. Is there a residual set $X_0 \subseteq X$ such that for each $x \in X_0$

$$\overline{\{T^p x : n \in \mathbb{P}\}} = X?$$

Thank You!

arXiv:2006.12385