Explicit construction of a multiplicatively circle-normal number

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Expanding Dynamics
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Introduction

- A usual discrete dynamical system involves iterating a map.

$$f, \qquad f^2 = f \circ f, \qquad f^3, \qquad f^4, \qquad \dots$$

- Because $f^n(f^m(x)) = f^{n+m}(x)$, this can be thought of as an action by the semigroup $(\mathbb{N}, +)$.

$$(n, x) \quad \longmapsto \quad f^n(x)$$

- What happens if we look at an action by $(\mathbb{N}, \times)$ instead?

$$(n, x) \quad \longmapsto \quad \textbf{?}$$

Expanding maps

Let $S^1 = [0,1]/\sim$ be the circle and denote by $E_m : S^1 \to S^1$ the map

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- Iterating E_m gives an $(\mathbb{N},+)$ action on S^1 **for each m**.

$$(n,x) \quad \longmapsto \quad m^n x \bmod 1.$$

  - Orbit: $\{\, m^n x \bmod 1 \,:\, n \in \mathbb{N} \,\}$.
  - Ergodic with respect to Lebesgue measure.
  - Almost every point is generic for Lebesgue (these numbers are called normal in base m).
  - Example: binary Champernowne number is normal in base 2.

$$0.\underline{1}\,\underline{10}\,\underline{11}\,\underline{100}\,\underline{101}\,\underline{110}\ldots$$

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- Because $E_n \circ E_m = E_{n\times m}$, we can also consider the action of $(\mathbb{N}, \times)$ on S^1 given by

$$(n, x) \quad \longmapsto \quad E_n(x).$$

  - Now the "orbit" of a point $x \in S^1$ is the set

$$\{\, nx \bmod 1 \,:\, n \in \mathbb{N} \,\}\,.$$

This is the classical ($(\mathbb{N}, +)$-action) orbit of 0 under rotation by x. It is equidistributed iff x is irrational.

Is this equidistributed **as an $(\mathbb{N}, \times)$ orbit**? What is the right definition?

Generic points

For $(\mathbb{N}, +)$ actions (iteration), a point x has an equidistributed orbit if

$$\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_A(f^i x) \to \mu(A) \quad \text{as } n \to \infty$$

for each interval A (here μ is Lebesgue).

Generic points

For $(\mathbb{N},+)$ actions (iteration), a point x is generic for μ if

$$\frac{1}{n}\sum_{i=1}^{n}\phi(f^i x) \to \int \phi \, \mathrm{d}\mu \quad \text{as } n \to \infty$$

for each continuous real-valued function ϕ.

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Definition

A point x is (F_n)-generic for μ under the G-action $g \star x$ if

$$\frac{1}{|F_n|}\sum_{g\in F_n}\phi(g\star x)\to\int\phi\,\mathrm{d}\mu \quad \text{as } n\to\infty$$

for each continuous ϕ, where $F_1, F_2, F_3, \ldots$ is a sequence of finite subsets of G called a "Følner sequence".

Følner sequences

Definition

A sequence (F_n) of finite subsets of G is a Følner sequence if

$$\lim_{n\to\infty} |gF_n \cap F_n| / |F_n| = 1 \quad \forall\ g \in G.$$

- For $(\mathbb{N}, +)$, the most common Følner sequence is $F_n = \{1, 2, \ldots, n\}$.

Definition

A Følner sequence (F_n) **in $(\mathbb{N}, \times)$** is called nice if

1. $F_n \subset F_{n+1}$,
2. each F_n is the set of divisors of some $L_n \in \mathbb{N}$,
3. for any $M \in \mathbb{N}$, eventually all L_n are multiples of M.

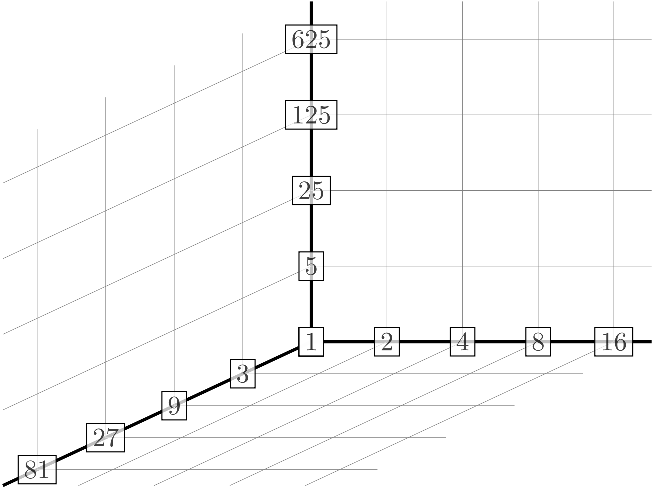
- For $(\mathbb{N}, \times)$, one example of a nice Følner sequence is

$$F_n = \{\, k \in \mathbb{N} : k \mid n! \,\}.$$

Other examples come from modeling $(\mathbb{N}, \times)$ as $\bigoplus_{p\in\mathbb{P}}(\mathbb{N}\cup\{0\}, +)$ via

$$2^{k_1} \cdot 3^{k_2} \cdot 5^{k_3} \cdot \dots \cdot p_r^{k_r} \quad \leftrightarrow \quad (k_1, k_2, \ldots, k_r, 0, 0, \ldots).$$

Geometric model



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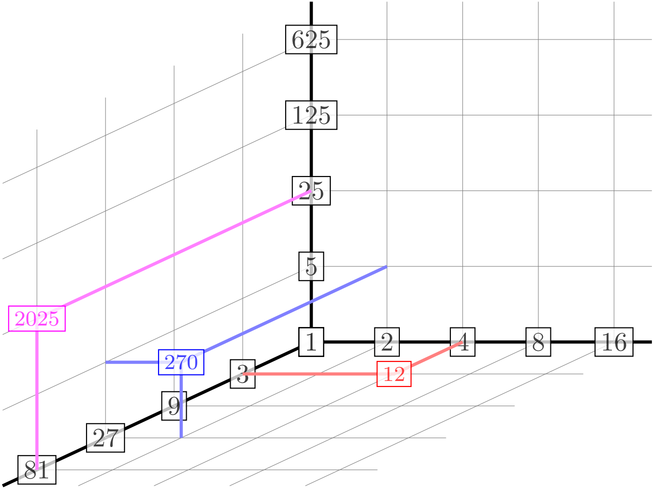
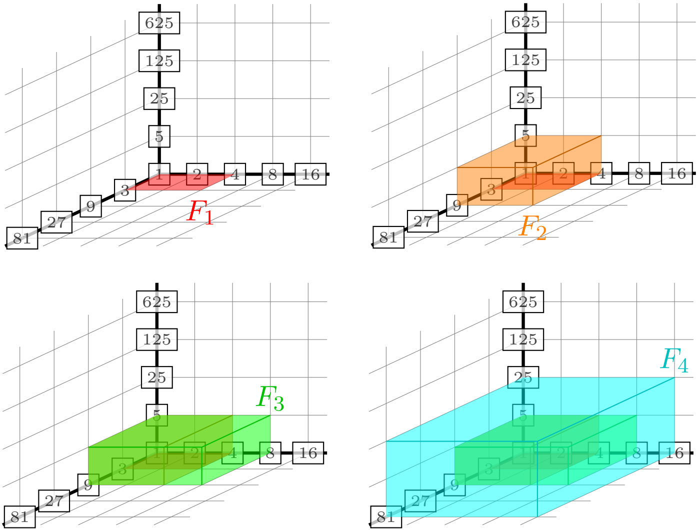


Example nice Følner sequence



Numbers → sequences

There are multiple ways we can encode $x \in S^1$ as a sequence.

- Binary, i.e., coding with 0 for $2^n x \in [0, \frac{1}{2})$ and 1 for $2^n x \in [\frac{1}{2}, 1)$.

- Coding 0 for $nx \in [0, \frac{1}{2})$ and 1 for $nx \in [\frac{1}{2}, 1)$.

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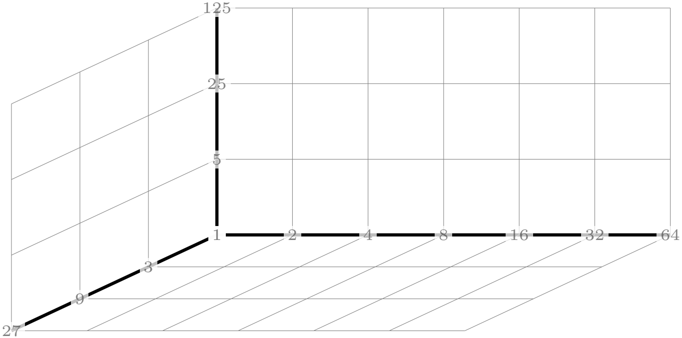
- Binary, i.e., coding with 0 for $2^n x \in [0, \frac{1}{2})$ and 1 for $2^n x \in [\frac{1}{2}, 1)$.
  - Omitting infinite tails of 1s, sequences and points are in bijection.
  - Lebesgue measure on S^1 ⇔ Bernoulli measure on $\{0, 1\}^{\mathbb{N}}$.
    - A number is normal to base 2 if the sequence is (F_n)-generic for Bernoulli under this $(\mathbb{N}, +)$-action.
  - The map $x \mapsto 2x \mod 1$ corresponds to a shift in this sequence.
- Coding 0 for $nx \in [0, \frac{1}{2})$ and 1 for $nx \in [\frac{1}{2}, 1)$.
  - The set of admissible sequences is now much smaller than $\{0, 1\}^{\mathbb{N}}$.
  - Lebesgue on S^1 ⇔ some unnamed measure on $\Lambda \subset \{0, 1\}^{\mathbb{N}}$.
    - A number is (F_n)-circle-normal if the sequence is (F_n)-generic for this measure under the $(\mathbb{N}, \times)$-action.
  - **The digits in positions 2^k fully determine x.**

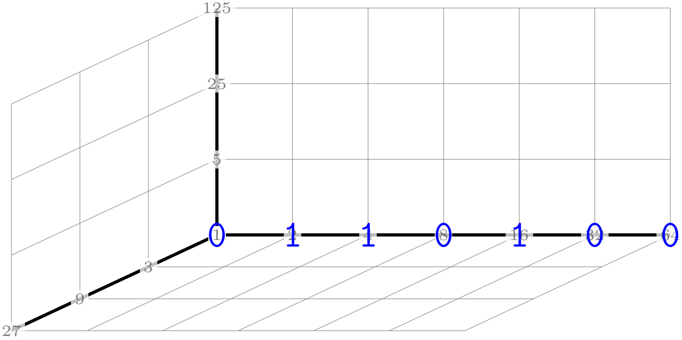
Open problems

We know almost every number is (F_n)-circle-normal for the $(\mathbb{N}, \times)$-action $(n, x) \mapsto nx \bmod 1$, but explicit examples are unknown.

1. Find an explicit nice (F_n) and an explicit $x \in S^1$ such that x is (F_n)-circle-normal.
2. Fix $F_n = \{\, k : k \mid n! \,\}$ and find an explicit $x \in S^1$ such that x is (F_n)-circle-normal.
3. Does there exist an x that is circle-normal for every nice Følner sequence? (This is true for the full shift.)
4. If x is circle-normal for every nice Følner sequence, does this imply other properties of x?



$$\{\overset{1}{\mathtt{X}}, \overset{2}{\mathtt{X}}, \overset{3}{\mathtt{X}}, \overset{4}{\mathtt{X}}, \overset{5}{\mathtt{X}}, \overset{6}{\mathtt{X}}, \overset{7}{\mathtt{X}}, \overset{8}{\mathtt{X}}, \overset{9}{\mathtt{X}}, \overset{10}{\mathtt{X}}, \ldots\}$$



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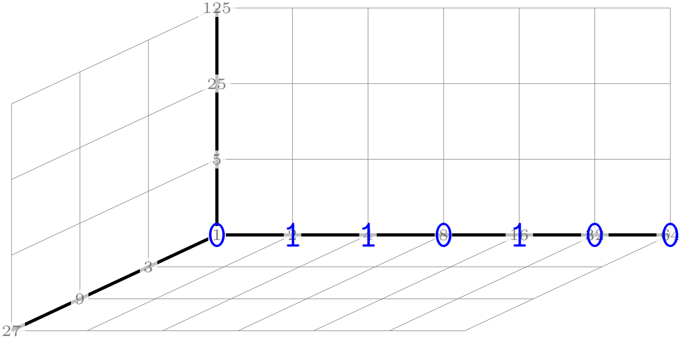
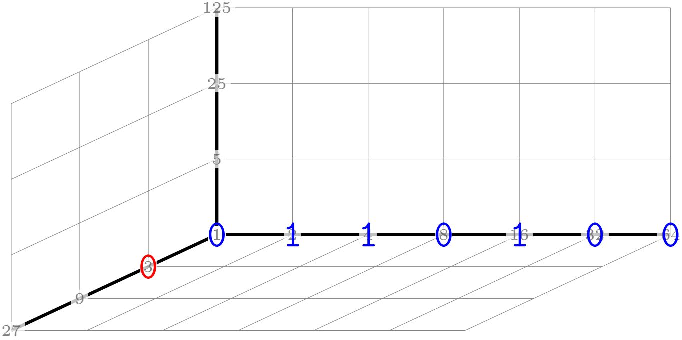


$$0.0110100\ldots \quad \leftrightarrow \quad \{\overset{1}{\color{blue}{0}}, \overset{2}{\color{blue}{1}}, \overset{3}{\mathtt{X}}, \overset{4}{\color{blue}{1}}, \overset{5}{\mathtt{X}}, \overset{6}{\mathtt{X}}, \overset{7}{\mathtt{X}}, \overset{8}{\color{blue}{0}}, \overset{9}{\mathtt{X}}, \overset{10}{\mathtt{X}}, \ldots\}$$



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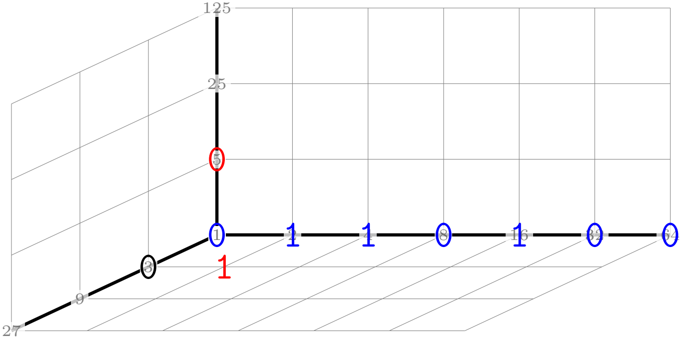
We know $x \in S^1$, so we know $3x \bmod 1$.



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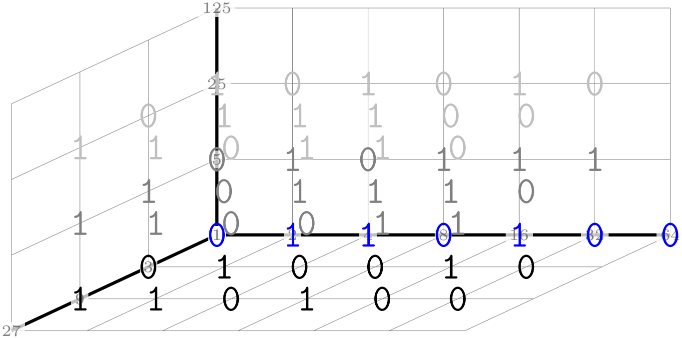
We know $x \in S^1$, so we know $nx \bmod 1$.



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