Phase Transition(s) for matrix equilibrium states

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The Problem

Question

When do matrix equilibrium states exhibit phase transitions?

- We have a number of examples of phase transitions in classical thermodynamic formalism.
- We would like examples of phase transitions in sub-additive thermodynamic formalism.
- We would like a better understanding of phase transitions overall.

Equilibrium/Gibbs states

Let $\Sigma = \{0, \ldots, M-1\}$ and $\sigma : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ be the shift map.

- "Scalar" equilibrium/Gibbs states. Let $\varphi : \Sigma^{\mathbb{N}} \to \mathbb{R}$ and $\beta \in \mathbb{R}$.
  - Equilibrium: Maximize

$$h_\mu(\sigma) + \beta \int \varphi d\mu$$

    over σ invariant measures.
  - Gibbs: There exist constants $C > 0$ and P such that

$$C^{-1} e^{-nP + \beta S_n \varphi(x)} \le \mu_\varphi[x_0 x_1 \cdots x_{n-1}] \le C e^{-nP + \beta S_n \varphi(x)}$$

    for all $x \in \Sigma^{\mathbb{N}}$.
- "Matrix" equilibrium/Gibbs states.
  - $\varphi : \Sigma^{\mathbb{N}} \to \mathbb{R}$ is replaced by $\mathcal{A} : \Sigma^{\mathbb{N}} \to M_d(\mathbb{R})$.
  - $S_n \varphi(x)$ is replaced by $\log \left\| \mathcal{A}(\sigma^{n-1} x) \cdots \mathcal{A}(\sigma x) \mathcal{A}(x) \right\|$.

Equilibrium/Gibbs states

Let $\mathcal{A} : \Sigma^{\mathbb{N}} \to M_d(\mathbb{R})$ depend only on 1-coordinate, $\mathcal{A} = (A_0, A_1, \ldots, A_{M-1})$ and $\beta \in \mathbb{R}$.

- Equilibrium: Maximize

$$h_\mu(\sigma) + \beta \underbrace{\lim_{n\to\infty} \frac{1}{n} \int \log \|A_{x_{n-1}} \cdots A_{x_1} A_{x_0}\| \, d\mu}_{\Lambda(\mathcal{A},\mu)}$$

  over σ invariant measures.

- Gibbs: There exist constants $C > 0$ and P such that

$$C^{-1} e^{-nP} \|A_{x_{n-1}} \cdots A_{x_0}\|^\beta \le \mu_{\mathcal{A},\beta}[x_0 \cdots x_{n-1}] \le C e^{-nP} \|A_{x_{n-1}} \cdots A_{x_0}\|^\beta$$

  for all $x \in \Sigma^{\mathbb{N}}$.

Pressure Functions

"Scalar" equilibrium/Gibbs states. Let $\varphi : \Sigma^{\mathbb{N}} \to \mathbb{R}$.

$$P_\varphi(\beta) = \sup\left\{h_\mu(\sigma) + \beta \int \varphi d\mu : \mu \text{ is a } \sigma \text{ invariant probability measure}\right\}$$
$$= \lim_{n\to\infty} \frac{1}{n} \log \sup_{x \in \Sigma^{\mathbb{N}}} \left(\sum_{|I|=n} e^{\beta S_n \varphi(Ix)} \right)$$

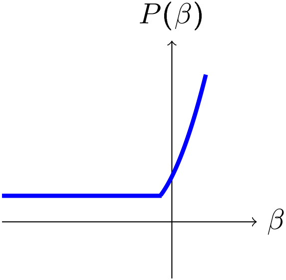
"Matrix" equilibrium/Gibbs states. Let $\mathcal{A} = (A_0, A_1, \dots, A_{M-1})$.

$$P_{\mathcal{A}}(\beta) = \sup\left\{h_\mu(\sigma) + \beta \Lambda(\mathcal{A}, \mu) : \mu \text{ is a } \sigma \text{ invariant probability measure}\right\}$$
$$= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{|I|=n} \|A_{i_{n-1}} \cdots A_{i_1} A_{i_0}\|^\beta \right)$$

Phase transitions

Definition

We say that there is a phase transition at β_0 if the pressure is not differentiable at β_0.



$$P'(\beta_0+) = \sup\left\{\int \varphi d\mu : \mu \text{ an eq. state for } \beta_0\varphi\right\}$$

$$P'(\beta_0-) = \inf\left\{\int \varphi d\mu : \mu \text{ an eq. state for } \beta_0\varphi\right\}$$

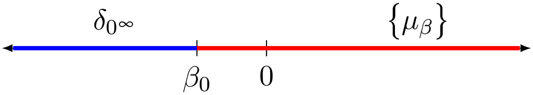
In particular non-differentiability of the pressure implies multiple equilibrium states.

Examples of Phase Transitions

Hofbauer potential:

$$\varphi(x) = \begin{cases} a_n & x \in [0^n 1] \\ \lim_{n\to\infty} a_n & x = 0^\infty \end{cases}$$

For carefully chosen sequences $\{a_n\}$ this potential has the following phase diagram:



Where μ_β is fully supported and has positive entropy for each β.

The Problem

Question

When do matrix equilibrium states exhibit phase transitions?

Question

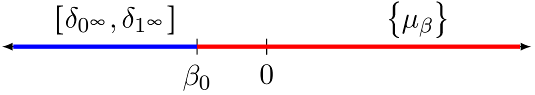
Does the collection

$$\mathcal{A} = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right)$$

have a phase transition? If yes describe the measures on either side of the transition.

Speculation

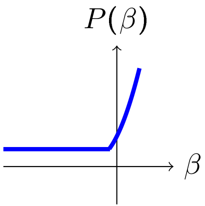
Yes, and the phase diagram probably looks like this:



- Where each μ_β is fully supported and has positive entropy.
- There are Hofbauer potentials which exhibit a similar kind of phase transition.

Some sub-Questions

Let $\mathcal{A} : \Sigma^{\mathbb{N}} \to GL_d(\mathbb{R})$ be locally constant (maybe depending on more then 1 coordinate).



For $\beta < \beta_0$

$$P(\beta) = a\beta + b.$$

This is sometimes called a "freezing transition".

Question

Assume $\mathcal{A}$ undergoes a freezing transition. Are the equilibrium states for $\beta < \beta_0$ supported on periodic points? Zero entropy?

Question

Can $\mathcal{A}$ undergo a phase transition which is not a freezing transition? Can $\mathcal{A}$ undergo multiple phase transitions?