The Density Polynomial Hales-Jewett (DPHJ) Problem

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- Outline:
  - Ramsey theory context and warmup
  - Statement of DPHJ
  - Simpler related open problems
  - References

Ramsey theory context

– Context matrix

	Coloring	Density
Linear	Hales-Jewett Theorem [HJ '63] IP van der Waerden [FW '78] van der Waerden [vdW '27]	Density Hales-Jewett [FK '91] IP Szemerédi Theorem [FK '85] Multidimensional Szemerédi [FK '78] Szemerédi's Theorem [Sz '75]
Polynomial	Polynomial Hales-Jewett [BL '99] (IP) Polynomial van der Waerden [BL '96]	Density Polynomial Hales-Jewett Problem IP Polynomial Szemerédi [BMc '98] Polynomial Szemerédi [BL '96] IP Furstenberg-Sárközy [BFMc '96] Furstenberg-Sárközy Theorem [F '77, S '78]

Ramsey theory warmup

– A special case of the multidimensional **van der Waerden theorem**: in any finite coloring of $\mathbb{N}^3$, there is a monochromatic simplex:

$$\begin{array}{l}
(\,a\,,\ b\,,\ c\,)\\
(\,a+n\,,\ b\,,\ c\,)\\
(\,a\,,\ b+n\,,\ c\,)\\
(\,a\,,\ b\,,\ c+n\,)
\end{array}$$

Ramsey theory warmup

– A special case of the multidimensional **polynomial van der Waerden theorem**: in any finite coloring of $\mathbb{N}^3$, there is a monochromatic simplex:

$$\begin{array}{l} (\ a\ ,\ b\ ,\ c\ ) \\ (\ a+n^2\ ,\ b\ ,\ c\ ) \\ (\ a\ ,\ b+n^2\ ,\ c\ ) \\ (\ a\ ,\ b\ ,\ c+n^2\ ) \end{array}$$

Ramsey theory warmup

– A special case of the multidimensional **Szemerédi theorem**: in any subset of $\mathbb{N}^3$ of positive density, there is a simplex:

$$\begin{array}{l} (a, b, c) \\ (a+n, b, c) \\ (a, b+n, c) \\ (a, b, c+n) \end{array}$$

– A special case of the multidimensional **polynomial Szemerédi theorem**: in any subset of $\mathbb{N}^3$ of positive density, there is a simplex:

$$\begin{array}{l} (a, b, c) \\ (a+n^2, b, c) \\ (a, b+n^2, c) \\ (a, b, c+n^2) \end{array}$$

Ramsey theory warmup

– For Hales-Jewett-type results, replace $\mathbb{N}$ with finite, non-empty subsets of $\mathbb{N}$:

$$\mathcal{P}_f(\mathbb{N}) = \left\{ F \subseteq \mathbb{N} \;\middle|\; F \text{ is finite and non-empty } \right\}.$$

– $A \subseteq \mathcal{P}_f(\mathbb{N})$ is **large** (of positive density) if

$$\limsup_{N \to \infty} \frac{\left|A \cap \mathcal{P}\big(\{1, \ldots, N\}\big)\right|}{\left|\mathcal{P}\big(\{1, \ldots, N\}\big)\right|} > 0.$$

Ramsey theory warmup

– A special case of the **Hales-Jewett theorem**: in any finite coloring of $\mathcal{P}_f(\mathbb{N})^3$, there is a monochromatic simplex:

$$\begin{array}{c|c} \begin{matrix} (\alpha, & \beta, & \gamma) \\ (\alpha \cup \nu, & \beta, & \gamma) \\ (\alpha, & \beta \cup \nu, & \gamma) \\ (\alpha, & \beta, & \gamma \cup \nu) \end{matrix} & \begin{matrix} (a, & b, & c) \\ (a + n, & b, & c) \\ (a, & b + n, & c) \\ (a, & b, & c + n) \end{matrix} \end{array}$$

where $\alpha, \beta, \gamma, \nu \in \mathcal{P}_f(\mathbb{N})$ and $\nu \cap (\alpha \cup \beta \cup \gamma) = \emptyset$

Ramsey theory warmup

– A quadratic special case of the **polynomial Hales-Jewett theorem**: in any finite coloring of $\mathcal{P}_f(\mathbb{N}^2)^3$, there is a monochromatic simplex:

$$\begin{array}{llll}
(\alpha & , \beta & , \gamma &) \\
(\alpha \cup (\nu \times \nu) & , \beta & , \gamma &) \\
(\alpha & , \beta \cup (\nu \times \nu) & , \gamma &) \\
(\alpha & , \beta & , \gamma \cup (\nu \times \nu) &)
\end{array}$$

where $\alpha, \beta, \gamma \in \mathcal{P}_f(\mathbb{N}^2)$, $\nu \in \mathcal{P}_f(\mathbb{N})$, and $(\nu \times \nu) \cap (\alpha \cup \beta \cup \gamma) = \emptyset$

Ramsey theory warmup

– A special case of the **density Hales-Jewett theorem**: in any large subset of $\mathcal{P}_f(\mathbb{N})^3$, there is a simplex:

$$
\begin{array}{c}
(\ \alpha\ ,\ \beta\ ,\ \gamma\ ) \\
(\ \alpha \cup \nu\ ,\ \beta\ ,\ \gamma\ ) \\
(\ \alpha\ ,\ \beta \cup \nu\ ,\ \gamma\ ) \\
(\ \alpha\ ,\ \beta\ ,\ \gamma \cup \nu\ )
\end{array}
$$

where $\alpha, \beta, \gamma, \nu \in \mathcal{P}_f(\mathbb{N})$ and $\nu \cap (\alpha \cup \beta \cup \gamma) = \emptyset$

Statement of DPHJ

– A special case of the **quadratic density Hales-Jewett problem**: in any large subset of $\mathcal{P}_f\big(\mathbb{N}^2\big)^3$, there is a simplex:

$$\begin{array}{llll}
(\; \alpha & , \; \beta & , \; \gamma &) \\
(\; \alpha \cup (\nu \times \nu) & , \; \beta & , \; \gamma &) \\
(\; \alpha & , \; \beta \cup (\nu \times \nu) & , \; \gamma &) \\
(\; \alpha & , \; \beta & , \; \gamma \cup (\nu \times \nu) &)
\end{array}$$

where $\alpha, \beta, \gamma \in \mathcal{P}_f\big(\mathbb{N}^2\big)$, $\nu \in \mathcal{P}_f\big(\mathbb{N}\big)$, and $(\nu \times \nu) \cap \big(\alpha \cup \beta \cup \gamma\big) = \emptyset$

– This is an open special case of the more-general **polynomial density Hales-Jewett** problem.

Related open problem: sets in a probability space

– Let (X, μ) be a probability space and A_1, A_2, ... be sets with measure $\mu(A_i) > \delta > 0$.

– **Pigeonhole principle**: There exists $i, n \in \mathbb{N}$ such that

$$\mu\big(A_i \cap A_{i+n}\big) > 0.$$

– Equivalent form of **Furstenberg [Fur77] & Sárközy [Sár78]**: There exists $i, n \in \mathbb{N}$ such that

$$\mu\big(A_i \cap A_{i+n^2}\big) > 0.$$

Related open problem: sets in a probability space

– Let (X, μ) be a probability space and A_1, A_2, … be sets with measure $\mu(A_i) > \delta > 0$.

– **Bergelson-Furstenberg-McCutcheon [BFM96]**: For all n_1, n_2, …, there exists i and distinct $j_1, \ldots, j_k \in \mathbb{N}$ such that

$$\mu\big(A_i \cap A_{i+(n_{j_1}+\cdots+n_{j_k})^2}\big) > 0.$$

– **Open problem**: There exists $N \in \mathbb{N}$ (independent of the A_i's) such that for all $n_1, \ldots, n_N \in \mathbb{N}$, there exists $i \in \mathbb{N}$ and distinct $j_1, \ldots, j_k \in \{1, \ldots, N\}$ such that

$$\mu\big(A_i \cap A_{i+(n_{j_1}+\cdots+n_{j_k})^2}\big) > 0.$$

Related open problem: large families of graphs

– Denote by G_N the family of graphs on the vertex set $\{1, \ldots, N\}$.

– For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all families of graphs $\mathcal{G} \subseteq G_N$ with $|\mathcal{G}| > \varepsilon |G_N|$, there exist $G_1, G_2 \in \mathcal{G}$ so that ...

– **Sperner's theorem:** ... $G_1 \subseteq G_2$.

– **McCutcheon [McC11]:** ... $G_1 \subseteq G_2$ and $G_2 \setminus G_1$ has a square number of edges.

– **Open problem [Gow09]:** ... $G_1 \subseteq G_2$ and $G_2 \setminus G_1$ is a union of a complete graph and isolated vertices.

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